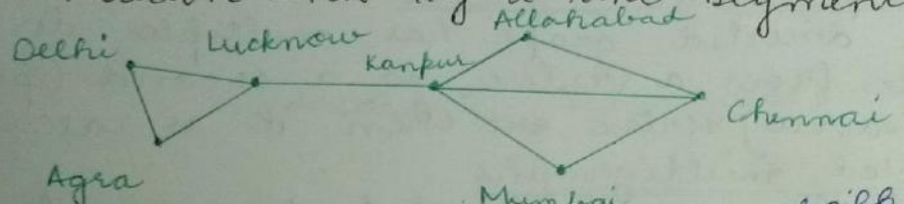
Graphs (Unit 4) ①

Def 1: A graph $G = (V, E)$ consists of V, a non-empty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints. Suppose a network is made up of data centres and communication links between computers. We can represent the location of each data centre by a point and each communication link by a line segment, like



Remark: The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set is called infinite graph. A graph with a finite vertex set is called a finite graph.

Here, in this graph, no edge connects a vertex to itself. Furthermore, no two different edges connect the same pair of vertices.

Def 2: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Def 3: Graphs that may have multiple edges connecting the same vertices are called multigraphs.

Remark: 1) An edge connecting a vertex to itself is called a loop.

2) Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices are called pseudographs.

3) When directions are not assigned to the edges of a graph, it is called undirected graph.

Def 1: A directed graph (or digraph) G(V, E) consists of a non empty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.

If a directed graph has multiple directed edge from a vertex to a second (possibly the same) vertex ~~are~~ then it is called directed multigraphs.

When there are m directed edges, each associated to an ordered pair of vertices (u, v) we say that (u, v) is an edge of multiplicity m.

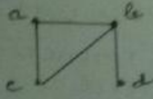
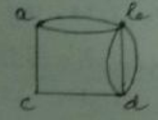
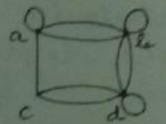
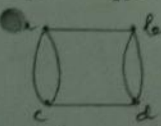
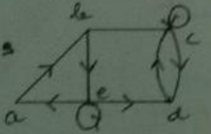
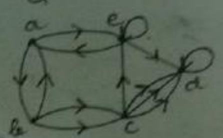
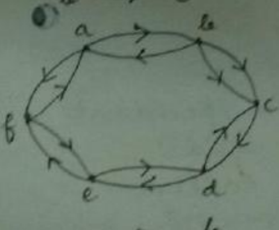
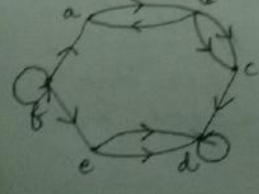
Remark: A graph with both directed and undirected edges is called a mixed graph.

Graph terminology	Edges	Multiple edges allowed?	Loops Allowed?
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multi graph	Directed	Yes	Yes
Mixed graph	Directed & undirected	Yes	Yes

Round Robin Tournament: is where each team plays each other team exactly once is called a round robin tournament. Here (a, b) represents an edge where team a beats team b.

Here team 1 is undefeated & Team 3 is winless.

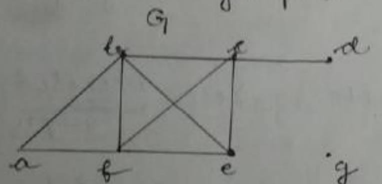
[Dete]rmine whether the graph has directed or [un]directed edges, whether it has multiple [ed]ges, whether it has one or more loops. Determine the type of graph. (3)

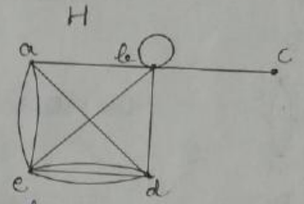
	Directed / Undirected	Multiple edges	Loops	Type
	Undirected	No	No	Simple graph
	Undirected	Yes	~~Yes~~ No	Multi ~~Pseudo~~ graph
	Undirected	Yes	Yes	Pseudo graph
	Undirected	Yes	No	Multi graph
	Directed	No	Yes	Directed ~~[illegible]~~ graph
	Directed	Yes	Yes	Directed multi graph
	Directed	No	No	Simple directed graph
	Directed	Yes	Yes	Directed multi graph

Def 5: Two vertices u and v in an (4) un-
graph G are called adjacent (or neighbo-
in G if u and v are endpoints of an
edge of G. If e is associated with {u, v
the edge e is called incident with the verti-
u and v. The edge e is also ~~called~~ said to connect
u and v. The vertices u and v are called
endpoints of an edge associated with {u, v}.

Def 6: The degree of a vertex in an undirected
graph is the number of edges incident with it
except that a loop at a vertex contributes
twice to the degree of that vertex. The degree
of the vertex is denoted by deg( )

Q What are the degrees of the vertices
in the graphs



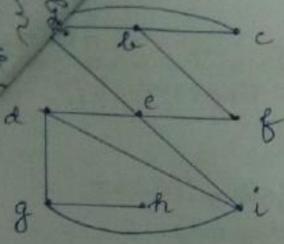


deg(a) = 2
deg(b) = 4
deg(c) = 4
deg(d) = 1
deg(e) = 3
deg(f) = 4
deg(g) = 0
vertex g is isolated
Vertex d is pendant

deg(a) = 4
deg(b) = 6
deg(c) = 1
deg(d) = 5
deg(e) = 6

Vertex c is pendant.

Remark: A vertex of degree zero is called
isolated. An isolated vertex is not adjacent
to any vertex.
A vertex is pendant iff it has degree one.
A pendant vertex is adjacent to exactly one
other vertex



Which vertices in this graph (5) are pendant and which are isolated ?

h is pendant vertex

$\because deg(h) = 1$

No vertex is isolated

$\because$ deg of any vertex $\neq 0$.

<u>The Handshaking (Lemma) theorem</u>

Let $G = (V, E)$ be an undirected graph with e edges Then $2e = \sum_{u \in V} deg(u)$

<u>Q</u> How many edges are there in a graph with 10 vertices each of degree 6.

<u>Sol</u> Here, sum of degrees = $6.10 = 60$.

By Handshaking theorem,

$$2e = 6 \cdot 10 = 60$$
$$e = 30.$$

<u>Note</u>: Sum of degrees of vertices of an undirected graph is even.

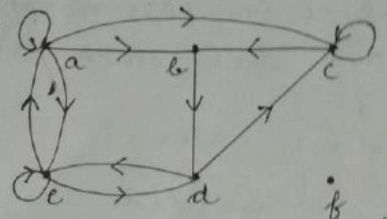
<u>Theorem</u>: An undirected graph has an even number of vertices of odd degree

<u>Def 7</u>: When (u, v) is an edge of the graph G with directed edges, u is said to be <u>adjacent to</u> v and v is said to be <u>adjacent from</u> u. The vertex u is called <u>initial vertex</u> of (u, v) and v is called <u>terminal</u> or <u>end vertex</u> of (u, v). The initial & terminal vertex of a loop are the same.

<u>Def 8</u>: In a graph with directed edges the <u>in-degree</u> of a vertex v, denoted by $deg^-(v)$ is the no of edges with v as their terminal vertex. The <u>out-degree</u> of v denoted by $deg^+(v)$ is the no of edges with v as their initial vertex

<u>Note</u>: A loop of a vertex contributes 1 to both the in-degree and the out-degree of the vertex

Q



Find the in-degree and out-degree of each vertex.

In-degrees

$deg^-(a) = 2$, $deg^-(b) = 2$

$deg^-(c) = 3$, $deg^-(d) = 2$

$deg^-(e) = 3$, $deg^-(f) = 0$

Out-degrees

$deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$

$deg^+(e) = 3$, $deg^+(f) = 0$

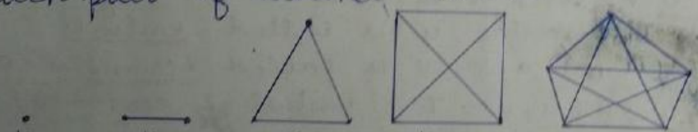
Note: Since each edge has an initial vertex and a terminal vertex, the sum of the in-degrees and the sum of out-degrees of all vertices in a graph with directed edges are the same. Both of these are the number of edges in the graph.

Theorem: Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

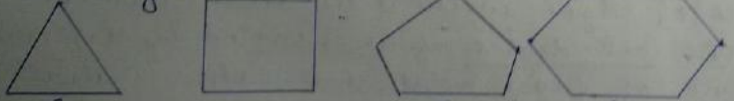
Special Simple graphs

Complete graphs: of n vertices, K_n is the simple graph that contains exactly one edge between each pair of distinct vertices



K_1 K_2 K_3 K_4 K_5

Cycles: C_n, $n \geq 3$ consists of n vertices $1, 2, \ldots n$ and edges $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$ and $\{n, 1\}$.



C_3 C_4 C_5 C_6.

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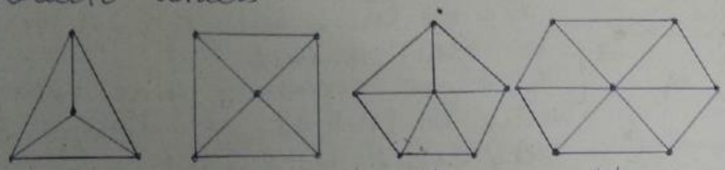
Wheels: If we add an additional vertex to C_n, $n \geq 3$ and connect this new vertex to each of the n vertices in C_n by new edges, we obtain wheels.

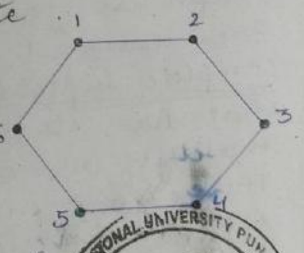


W_3 W_4 W_5 W_6

Bipartite graphs

Def 9: A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (no edge in G connects either two vertices in V_1 or two vertices in V_2). In this case, (V_1, V_2) is called bipartition of the vertex set V of G.

Eg Show that C_6 is bipartite.



C_6

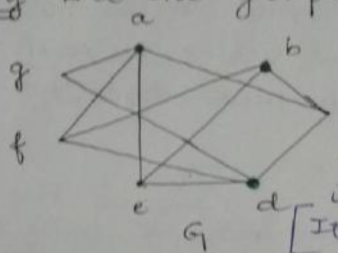
Hence, the vertex set can be partitioned into the two sets $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

Eg K_3 is not bipartite.

K_3

If we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If bipartite these two vertices can't be connected by an edge but in K_3, each vertex is connected to every other vertex by an edge.

Eg Are the graphs G & H bipartite?

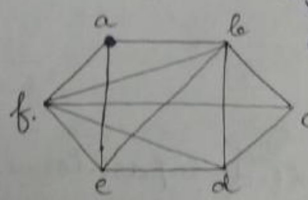


G

G is bipartite because vertex set is union of disjoint sets $\{a, b, d\}$ & $\{c, e, f, g\}$. Each edge connects a vertex in one of these subsets to a vertex in other subset.

[It is not necessary that every vertex in $\{a, b, d\}$ is adjacent to every vertex in $\{c, e, f, g\}$

Eg. b and g are not adjacent]



H

In graph H,
a is connected to f, e, b
so f, e, b should be in a different set of vertices to a
but also, f and b are adjacent which is not possible in bipartite graph. Hence, H is not bipartite.

Theorem: A simple graph is bipartite iff it is possible to assign one of two different colours to each vertex of the graph so that no two adjacent vertices are assigned the same color.

complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices iff one vertex is in the first subset & the other vertex is in the second subset.

$K_{2,3}$

$K_{3,3}$

$K_{2,6}$

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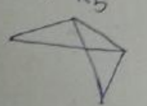
Def 10 A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \neq G$

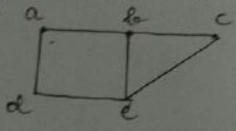
K_5

subgraph of K_5

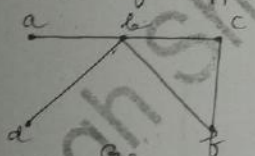


Def 11. The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 & G_2 is denoted by $G_1 \cup G_2$.

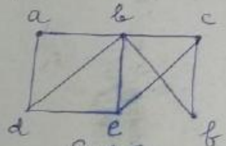
eg Find the union of G_1 & G_2



G_1



G_2



$G_1 \cup G_2$

Degree sequence : of a graph is the sequence of the degrees of the vertices of the graph in non increasing order.

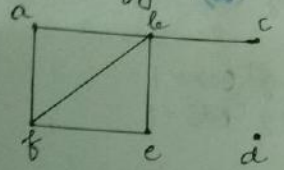
Regular A simple graph is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if every vertex in the graph has degree n

Complementary graph $\bar{G}$ of a simple graph G has the same vertices of G. Two ~~do~~ vertices are adjacent in $\bar{G}$ iff they are not adjacent in G

Exercise

(10)

Find the no of vertices, no of edges and the degree of each vertex in the following graphs. Identify all isolated & pendant vertices. 6

1) 

Vertices → 6 , edges → 6

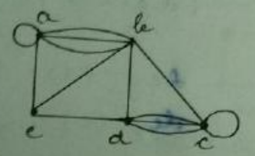
deg(a) = 2   deg(d) = 0

deg(b) = 4   deg(e) = 2

deg(c) = 1   deg(f) = 3

Isolated → d , Pendant → c

deg sequence → 4, 3, 2, 2, 1, 0

2) 

Vertices → 5   edges → 13

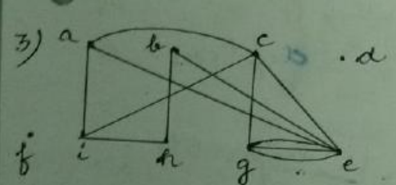
deg(a) = 6   deg(c) = 6

deg(b) = 6   deg(d) = 5

deg(e) = 3

No isolated, No pendant

Deg seq → 6, 6, 6, 5, 3

3) 

Vertices → 9   edges → 12

deg(a) = 3.   deg(f) = 0

deg(b) = 2.   deg(g) = 4.

deg(c) = 4.   deg(h) = 2.

deg(d) = 0   deg(i) = 3.

deg(e) = 6.

d and f are isolated , No pendant

Deg seq → 6, 4, 4, 3, 3, 2, 2, 0, 0

5) Can a simple graph exist with 15 vertices each of degree 5 ?

Let the simple graph exist

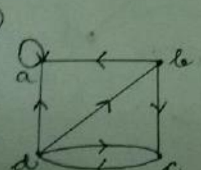
then by handshaking theorem

$2e = 15 \cdot 5 = 75$

$e = 75/2$ which is not possible

∴ Such a simple graph is not possible.

Determine the no of vertices and edges and find the in-degree and out-degree of each vertex for given directed multigraph.

7) 

Vertices → 4

Edges → 7

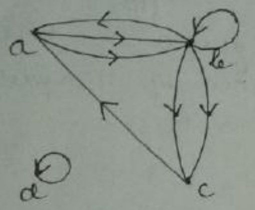
In degrees

$deg^-(a) = 3$, $deg^-(b) = 1$

$deg^-(c) = 2$, $deg^-(d) = 1$

Out - degrees

$deg^+(a) = 1$ $deg^+(b) = 2$ $deg^+(c) = 1$ $deg^+(d) = 3$

8) 

Vertices → 4 Edges → 8

In-degrees

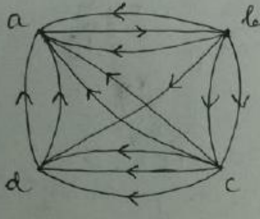
$deg^-(a) = 2$ $deg^-(c) = 2$

$deg^-(b) = 3$ $deg^-(d) = 1$

Out-degrees

$deg^+(a) = 2$ $deg^+(c) = 1$

$deg^+(b) = 4$ $deg^+(d) = 1$

9) 

Vertices → 5 Edges → 12

In-degrees

$deg^-(a) = 6$ $deg^-(d) = \cancel{3}\,4$

$deg^-(b) = 1$ $deg^-(e) = 0$

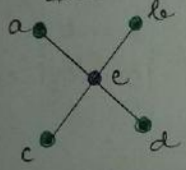
e $deg^-(c) = 2$

Out-degrees

$deg^+(a) = 1$ $deg^+(c) = 5$ $deg^+(e) = 0$

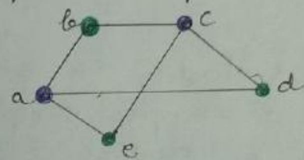
$deg^+(b) = 5$ $deg^+(d) = 2$

Determine whether the graph is bipartite.

21) 

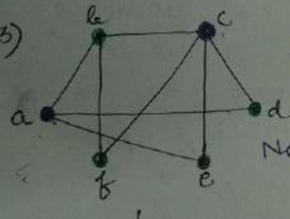
deg(a) = 1
deg(b) = 1
deg(c) = 1
deg(d) = 1
deg(e) = 4
deg seq → 4, 1, 1, 1, 1

Bipartite

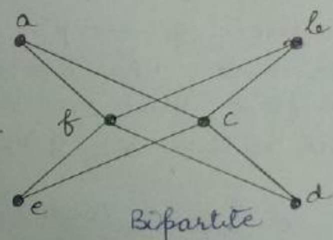
22) 

deg(a) = 3
deg(b) = 2
deg(c) = 3
deg(d) = 2
deg(e) = 2
deg seq
3, 3, 2, 2, 2

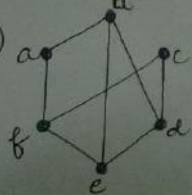
Bipartite

23) 

b and f are joined & have same color. Not bipartite

24) 

Bipartite

25) 

a and b are joined and have same color so not bipartite.

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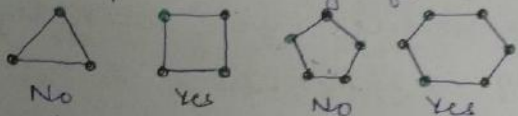
6) For which values of n are these graphs bipartite?

a) K_n


Yes No No

It is bipartite only for $n = 2$.

b) C_n


No Yes No Yes

It is bipartite for $n = 4, 6, 8, 10, \ldots$

c) W_n


No No No No

It is not bipartite for any $n \geq 3$

29) How many vertices and how many edges do these graphs have?

	Vertices	Edges
a) K_n	n	$\frac{n(n-1)}{2}$
b) C_n	n	n
c) W_n	$n+1$	$2n$
d) $K_{m,n}$	$m+n$	mn

In K_n, every vertex has degree $n-1$

so $\sum_{v \in V} \deg(v) = n(n-1) = 2e$

$\therefore e = \frac{n(n-1)}{2}$

In C_n, every vertex has degree 2

$\sum_{v \in V} \deg(v) = 2n = 2e$

$\Rightarrow e = n$

In W_n, every vertex (except one at the centre having degree n) has degree 3.

$\sum_{v \in V} \deg(v) = 3n + n = 4n = 2e$

$e = 2n$

In $K_{m,n}$, in the set of m vertices, every vertex has degree n & in set of n vertices, every vertex has degree m

(13)

$$\sum_{v \in V} deg(v) = mn + nm = 2mn = 2e$$
$$\Rightarrow e = mn$$

31) Find the degree sequence of

a) K_4.

There are 4 vertices each with degree 3

deg seq $\rightarrow 3, 3, 3, 3$

b) C_4

There are 4 vertices each with degree 2

deg seq $\rightarrow 2, 2, 2, 2$

c) W_4

There are 4 vertices each with deg 3 and one vertex with degree 4

deg seq $\rightarrow 4, 3, 3, 3, 3$

d) $K_{2,3}$

There are 2 vertices of deg 3 and 3 vertices of deg 2

$\therefore$ deg seq is $3, 3, 2, 2, 2$

32) What is degree sequence of bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

There are m vertices with degree n each & n vertices with degree m each.

If $m > n$ seq $\rightarrow \underbrace{m, m, \ldots m}_{(n \text{ times})}, \underbrace{n, n, \ldots, n}_{(m \text{ times})}$

If $m < n$ deg seq $\rightarrow \underbrace{n, n, \ldots n}_{(m \text{ times})}, \underbrace{m, m, \ldots m}_{(n \text{ times})}$

33) What is the degree sequence of K_n, where n is a positive integer?

Every vertex in K_n has degree $n-1$

$\therefore$ Deg seq is $\underbrace{n-1, n-1, \ldots, n-1}_{(n \text{ times})}$

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Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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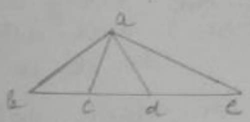
34) How many edges does a graph have its deg seq is 4, 3, 3, 2, 2 ? Draw such a graph.

$\sum_{v \in V} deg(v) = 4+3+3+2+2 = 14$

No. of edges = 7.



35) How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1 ? Draw such a graph

No. of ~~each~~ edges $= \frac{5+2+2+2+2+1}{2} = 7$

47) For which values of n are these graph regular ?

a) K_n Every vertex has degree $n-1$
So it is regular $\forall\ n \geq 1$

b) C_n Every vertex has degree 2
so it is regular $\forall\ n \geq 3$

c) W_n Every vertex (except 1 having degree n) has degree 3
so it is regular only for $n = 3$.

48) For which values of m and n is $K_{m,n}$ regular ?

m vertices are there with deg n
n vertices are there with deg m.
It is regular only when $m = n$

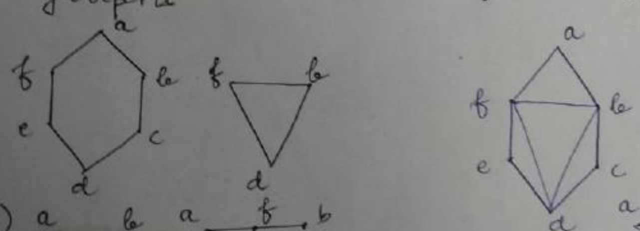
49) How many vertices does a regular graph of degree four with 10 edges have ?

let there be n vertices each of degree 4.
$\therefore$ By Handshaking theorem,

$$2e = \sum_{v \in V} \deg(v)$$

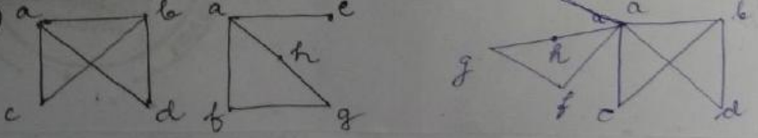
$$2 \times 10 = 4n \Rightarrow n = 5$$

50) Find the union of the given pair of graphs



51)

52)



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Note: I) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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54) If G is a simple graph with 15 edges and $\bar{G}$ has 13 edges, how many vertices does G have?

Let there be v vertices.

In total v can be adjacent to v−1 vertices

$\therefore$ No. of edges possible $= \frac{v(v-1)}{2}$

These edges will be either in G or in $\bar{G}$

$\therefore 15+13 = \frac{v(v-1)}{2}$

$28 \cdot 2 = v(v-1)$

$8 \cdot 7 = 56 = v(v-1)$

$v = 8$ Ans: 8 vertices

55) If a simple graph G has v vertices and e edges, how many edges does $\bar{G}$ have?

G has v vertices, there are v−1 vertices that can be adjacent to v.

So sum of degrees $= v(v-1)$

Total no. of edges possible $= \frac{v(v-1)}{2}$

Now if G has e edges

then $\bar{G}$ has $\left(\frac{v(v-1)}{2} - e\right)$ edges.

56) If the degree sequence of the simple graph G is 4, 3, 3, 2, 2 what is the degree seq of $\bar{G}$?

Here we have 5 vertices

so deg seq. 2, 2, 1, 1, 0

$4 \to 0$
$3 \to 1$
$3 \to 1$
$2 \to 2$
$2 \to 2$

Sum should be 4

57) If the deg seq of the simple graph G 17 is $d_1, d_2, \ldots, d_n$ what is the degree sequence of $\overline{G}$?

Total n vertices. Each vertex can max be connected to $n-1$ vertices (simple graph)

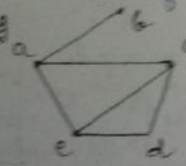
$\therefore$ deg seq $\rightarrow$

$n-1-d_n, n-1-d_{n-1}, \ldots, n-1-d_2, n-1-d_1$

Representing graphs and graph isomorphism

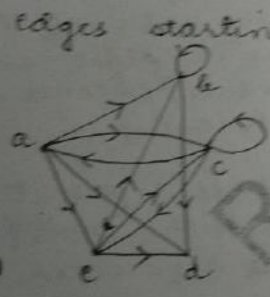
1) To list all the edges of the graph with no multiple edges
2) A graph with no multiple edges can also be represented by using adjacency lists which specify the vertices that are adjacent to each vertex of the graph

eg



Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

eg Represent the digraph by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph



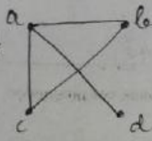
Adjacency list

Initial vertex	Terminal vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	–
e	b, c, d

Adjacency Matrix

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $1, 2, \ldots, n$. The adjacency matrix A (or A_G) of G with ~~resptes~~ respect to this listing of vertices is the $n \times n$ zero-one matrix with 1 as its (i,j)th entry when i and j are adjacent and 0 as its (i,j)th entry when they are not adjacent

$$a_{ij} = \begin{cases} 1 & \text{when } \{i,j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

eg 

Use adjacency matrix to represent the graph

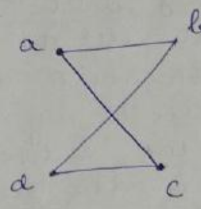
(19)

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

eg Draw a graph with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to ordering of vertices a, b, c, d.



Note: An adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there are $n!$ different matrices for a graph with n vertices because there are $n!$ different orderings of n vertices.

Note: The adjacency matrix of a simple graph is symmetric i.e $a_{ij} = a_{ji}$ because both of these entries are 1 when i and j are adjacent and both are 0 otherwise. Furthermore, because a simple graph has no loops, each entry a_{ii}, $i = 1, 2, \ldots n$ is 0.

Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. A loop at the vertex v_i is represented by a 1 at (i, i)th position of adjacency matrix. When multiple edges connecting the same pair of vertices v_i and v_j or multiple loops at the same vertex are present, then (i, j)th entry of matrix equals the no. of edges that are associated to $\{v_i, v_j\}$. All undirected graphs including multigraphs & pseudographs have symmetric adjacency matrices.

Eg Draw adjacency matrix using ordering a, b, c, d

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

The matrix for a directed graph $G = (V, E)$ has a 1 in its (i, j)th position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots, v_n$ is an arbitrary listing of vertices of directed graph.

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

The adjacency matrix need not be symmetric because there may be an edge from v_i to v_j but not from v_j to v_i

Q Draw adjacency matrix using the ordering a, b, c, d, e

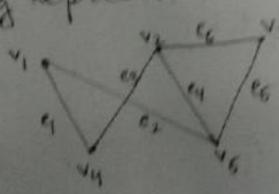
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Incidence Matrices

Let $G = (V, E)$ be an undirected graph. Suppose that $1, 2, \ldots, n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$ where

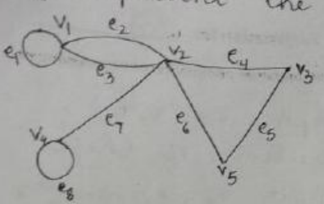
$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } i, \\ 0 & \text{otherwise} \end{cases}$$

Eg Represent the graph with incidence matrix



	e_1	e_2	e_3	e_4	e_5	e_6
1	1	1	0	0	0	0
2	0	0	1	1	0	1
3	0	0	0	0	1	1
4	1	0	1	0	0	0
5	0	1	0	1	1	0

eg Represent the graph using incidence mat



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	
1	1	1	1	0	0	0	0	
2	0	1	1	1	0	1	1	0
3	0	0	0	1	1	0	0	0
4	0	0	0	0	0	0	1	1
5	0	0	0	0	1	1	0	0

Isomorphism of graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2 $\forall\ a, b \in V_1$ such a function f is called an isomorphism.

Isomorphism of simple graphs is an equivalence relation.

Let $G_1 = (V_1, E_1)$

Reflexive G_1 is isomorphic to itself with isomorphism I (identity function)

If a and $b \in V_1$ s.t a & b are adjacent in G_1

then $I(a) = a$ & $I(b) = b$ are adjacent in G_1

Symmetric Let $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$

$G_1 \cong G_2$ and $f : V_1 \to V_2$ be an isomophism

$\Rightarrow f^{-1} : V_2 \to V_1$ is an isomorphism

$\Rightarrow G_2 \cong G_1$

Transitive Let $G_1 \cong G_2$ & $G_2 \cong G_3$.

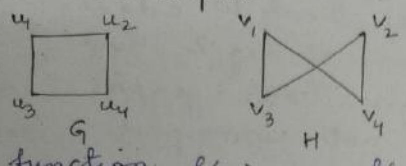
$\Rightarrow \exists$ a $f : V_1 \to V_2$ & $g : V_2 \to V_3$ s.t f & g are isomorphism

$\Rightarrow$ ~~$f \circ g$~~ $g \circ f : V_1 \to V_3$ is also an isomorphism

$\Rightarrow G_1 \cong G_3$.

$\therefore$ it is an equivalence relation.

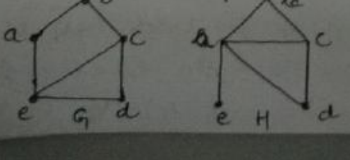
Eg Show that the graphs $G = (V, E)$ & $H = (W, F)$ are isomorphic



The function $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ is 1-1 onto map between V and W
To check that this correspondence preserves adjacency, we note adjacent vertices in G are u_1 & u_2, u_1 & u_3, u_2 & u_4 & u_3 & u_4 and each of the pairs $f(u_1) = v_1$ & $f(u_2) = v_4$, $f(u_1) = v_1$ & $f(u_3) = v_3$, $f(u_2) = v_4$ & $f(u_4) = v_2$, $f(u_3) = v_3$ & $f(u_4) = v_2$ are adjacent in H

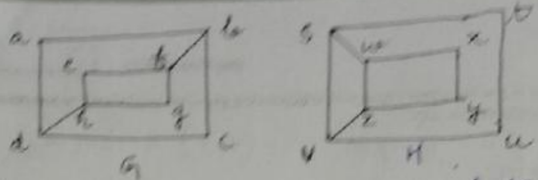
Note: Sometimes we can show that two graphs are not isomorphic if we can find a property only one of the two graphs has, but that is preserved by isomorphism. A property preserved by isomorphism of graphs is called a graph invariant.
For eg: 1) Two isomorphic simple graphs must have same no. of vertices
2) They have same no. of edges
3) They have same degrees i.e a vertex v of degree d in G must correspond to a vertex f(v) of degree d in H because a vertex w in G is adjacent to v iff f(v) & f(w) are adjacent in H.

Eg Determine whether the graphs are isomorphic.



Not isomorphic
H has a vertex of deg 4 which is not present in G.

Eg



Here, we have 8 vertices and 10 edges. Moreover 4 vertices are of deg 2 and 4 vertices of deg 3 in both graphs. But still they are not isomorphic.

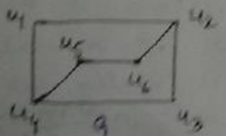
In G, $deg(a) = 2$

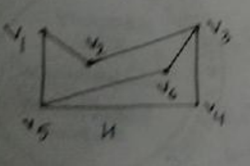
$\therefore$ a can correspond to t, u, x or y in H.

Now a is adjacent to 2 vertices of degree 3 each. which is not true for any of the four vertices t, u, x or y in H.

Note: To show that a function f from a vertex set of a graph G to the vertex set of graph H is an isomorphism, we need to show that f preserves the presence and absence of edges. We can show that adjacency matrix G and H are same, when rows & columns are labelled to correspond to the images under f of vertices in G that are the labels of these rows & columns in adjacency matrix of G

Eg Determine whether isomorphic or not





Both G & H has 6 vertices 7 edges. Both graphs have 4 vertices of degree 2 & 2 vertices of degree 3. Now, we try to find an isomorphism.

Here $deg(u_1) = 2$ and u_1 is adjacent to u_2 & u_4 each of degree 3. So $f(u_1) = v_6$ or v_4

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Let $f(u_1) = v_6$ (if it doesnot work, we try $f(u_1) = v_4$)

Now $f(u_4)$ = either v_5 or v_3

Let $f(u_4) = v_5$.

then $f(u_2) = v_3$.

Also $f(u_3) = v_4$

Now $f(u_5) = v_1$ $\because u_5$ is adjacent to u_4

& $f(u_6) = v_2$. $\therefore f(u_5)$ is adjacent to $f(u_4) = v_5$

& v_1 is adjacent to v_5.

To see whether f preserves edges, we draw adjacency matrix of G.

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix of H with rows & columns labeled by images of corresponding vertices in G

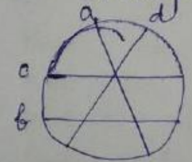
$$A_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$A_G = A_H$ $\therefore$ f preserves edges

$\therefore$ f is isomorphism

so G & H are isomorphic.

A circle graph is a graph whose vertices be associated with chords of a circle such that two vertices are adjacent only if the the corresponding chords in the circle intersect



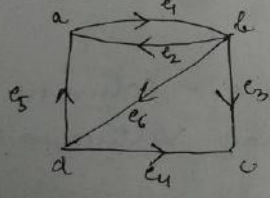
Adjacency Matrix

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

* If G and H are isomorphic simple graphs then $\overline{G}$ and $\overline{H}$ are also isomorphic
* A simple graph G is called self-complementary if G and $\overline{G}$ are isomorphic

Incidence matrix for a directed graph

$$a_{ij} = \begin{cases} 1 , & v_i \text{ is } \cancel{\text{adjacent}} \text{ incident out from } e_j \\ -1 , & v_i \text{ is } \text{——— in from } e_j \\ 0 , & v_i \text{ is not incident to } e_j \end{cases}$$



$$\begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

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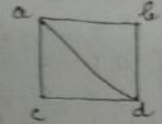
Exercises

1) 15)



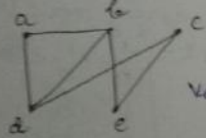
Adjacency list

Vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Adjacency Matrix

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2) & 6)



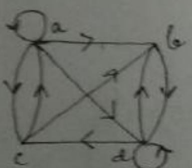
Adjacency list

Vertex	Adjacent vertices
a	b, d
b	a, d, e
c	d, e
d	a, b, c
e	b, c

Adjacency Matrix

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

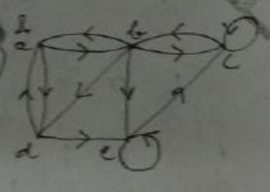
3) & 7)



Initial vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

4) & 8)

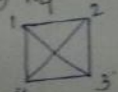


Initial vertex	Terminal vertices
a	b, d
b	a, c, d, e
c	b, c
d	a, e
e	c, e

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

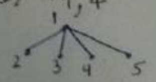
9) Represent by adjacency matrix

a) K_4



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b) $K_{1,4}$

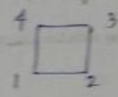


$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

c) $K_{2,3}$

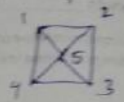
$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

d) C_4



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

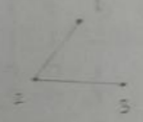
e) W_4



$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

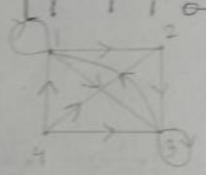
Draw a graph with given adjacency matrix

10) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$



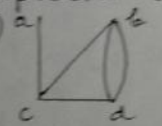
11) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

12) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$



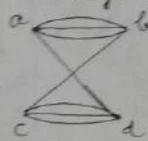
Represent using adjacency matrix

13)



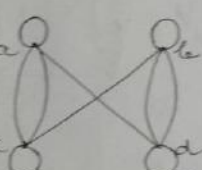
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

14)



$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

15)

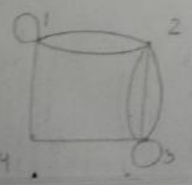


$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

Draw undirected graph represented by given adjacency matrix

16) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

17) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$



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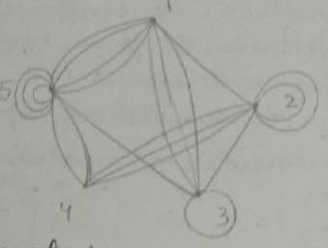
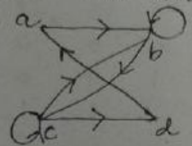
Note: I) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
II) Exchange of sheet will be considered as UMC.

18) $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$



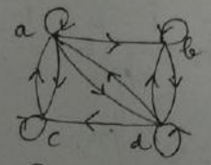
Find the adjacency matrix

19)



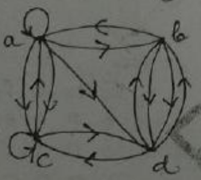
$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 1 \\ d & 1 & 0 & 0 & 0 \end{array}$

20)



$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 0 \\ d & 1 & 1 & 1 & 1 \end{array}$

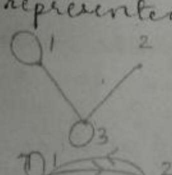
21)



$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 2 & 1 \\ b & 1 & 0 & 0 & 2 \\ c & 1 & 0 & 1 & 1 \\ d & 0 & 2 & 1 & 0 \end{array}$

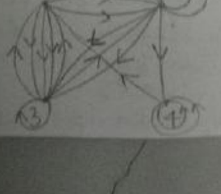
Draw the graph represented by given adjacency matrix.

22) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$



23) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

24) $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$



25) Is every zero-one square matrix that is symmetric & has zeros on diagonal the adjacency matrix of a simple graph?

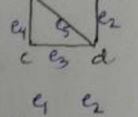
∵ symmetric ⇒ undirected

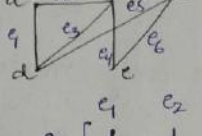
diag entries zero ⇒ No loops

zero-one entries ⇒ No multiple edges

∴ simple graph.

26) Write incidence matrix

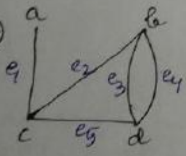


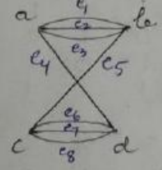


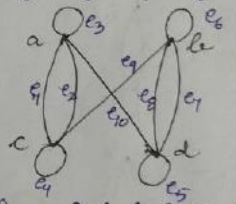
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

27)







$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & [illegible] & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

28) What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

Undirected → degree of vertices − no. of loops at vertex

Directed → out-degree. $deg^+(v)$

29) What is the sum of the entries in a column of the adjacency matrix for an undirected graph For a directed graph?

Undirected → degree of vertices − no of loops at vertex

Directed → in-degree $deg^-(v)$

30) What is the sum of the entries in a row of the incidence matrix for an undirected graph?

Ans $deg(v)$ − no. of loops at vertex v.

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Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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31) What is the sum of the entries in a column of the incidence matrix for an undirected graph ?

2, if edge is not a loop
1, if edge is a loop .

32) Find adjacency matrix of

a) K_n

$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & \cdots & v_{n-1} & v_n \\ \hline v_1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ v_2 & 1 & 0 & 1 & \cdots & 1 & 1 \\ v_3 & 1 & 1 & 0 & & 1 & 1 \\ \vdots & & & & & & \\ v_{n-1} & 1 & 1 & 1 & & 0 & 1 \\ v_n & 1 & 1 & 1 & & 1 & 0 \end{array}$$

b) C_n

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & \cdots & n-1 & n \\ \hline 1 & 0 & 1 & 0 & & 0 & 1 \\ 2 & 1 & 0 & 1 & & 0 & 0 \\ 3 & 0 & 1 & \cdot & 1 & 0 & 0 \\ \vdots & & & & & & \\ n-1 & 0 & 0 & 0 & 1 & 0 & 1 \\ n & 1 & 0 & 0 & & 1 & 0 \end{array}$$

c) W_n

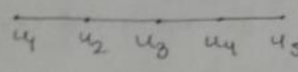
$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & \cdots & n-1 & n & n+1 \\ \hline 1 & 0 & 1 & 0 & & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & \cdot & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \vdots & & & & & & & \vdots \\ n-1 & 0 & 0 & 0 & & 0 & 1 & 1 \\ n & 1 & 0 & 0 & & 1 & 0 & 1 \\ n+1 & 1 & 1 & 1 & & 1 & 1 & 0 \end{array}$$

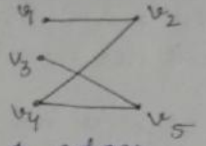
d) $K_{m,n}$

$$\begin{array}{c|cccccccc} & 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & m+n \\ \hline 1 & 0 & 0 & \cdots & 0 & 1 & 1 & & 1 \\ 2 & 0 & 0 & \cdots & 0 & 1 & 1 & & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m & 0 & 0 & \cdots & 0 & 1 & 1 & & 1 \\ m+1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ m+2 & 1 & 1 & \cdots & 1 & 0 & 0 & & 0 \\ \vdots & & & & & & & & \\ m+n & 1 & 1 & \cdots & 1 & 0 & 0 & & 0 \end{array}$$

Determine whether the following pairs of g are isomorphic or not. Exhibit an isomorph or provide an argument that none exists. (31)

34)





Both have 5 vertices & 4 edges.
There are 3 vertices of deg 2 each & 2 vertices of degree 1 each

$f(u_1) = v_1$ or v_3

Let $f(u_1) = v_1$

$\Rightarrow f(u_2) = v_2$

$f(u_3) = v_4$

$f(u_4) = v_5$

$f(u_5) = v_3$.

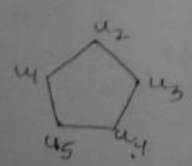
We can check edges are preserved

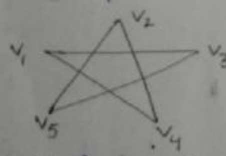
$$\begin{array}{c|c} & \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{array}{ccccc} v_1 & v_2 & v_4 & v_5 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_4 \\ v_5 \\ v_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore$ f is an isomorphism.

$\therefore$ graphs are isomorphic.

35)





There are 5 vertices & 5 edges. Each vertex has degree 2

Let $f(u_1) = v_1$

u_1 is adjacent to u_2 & u_5

v_1 ——— v_3 & v_4

Let $f(u_2) = v_3$ $\qquad f(u_3) = v_5$

$f(u_5) = v_4$

$f(u_4) = v_2$

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Registration No. 32

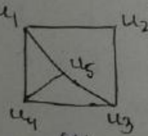
Note: I) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
II) Exchange of sheet will be considered as UMC.

$$\begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{matrix} v_1 & v_3 & v_5 & v_2 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_3 \\ v_5 \\ v_2 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

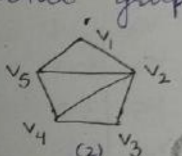
$\therefore$ f is isomorphism

Hence graphs are isomorphic

36)



(1)



(2)

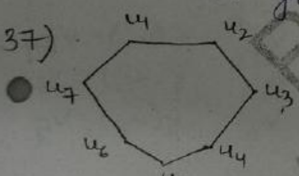
There are 5 vertices & 7 edges

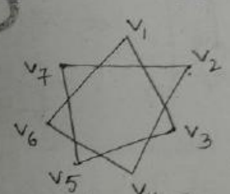
In graph (1), there is 1 vertex (u_2) with deg 2

In graph (2), there are 2 vertices (v_1 & v_3) with degree 2

$\therefore$ graphs are not isomorphic.

37)





There are 7 vertices & 7 edges

Each vertex is of degree 2

Let $f(u_1) = v_1$ $\quad f(u_5) = v_2$

$f(u_2) = v_3$ $\quad f(u_6) = v_4$

$f(u_3) = v_5$ $\quad f(u_7) = v_6$

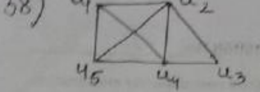
$f(u_4) = v_7$

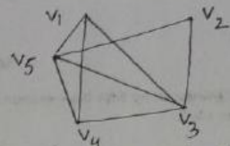
Now edges are preserved (adjacency matrix) by f

$\therefore$ f is isomorphism Graphs are isomorphic

33

38)





Both have 5 vertices & 8 edges

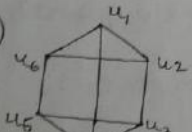
let $f(u_3) = v_2$
$f(u_2) = v_3$
$f(u_4) = v_5$ $\qquad$ f is 1-1 onto.
$f(u_1) = v_1$
$f(u_5) = v_4$

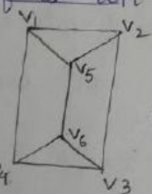
$$\begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{matrix} v_1 & v_3 & v_2 & v_5 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_3 \\ v_2 \\ v_5 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

edges are preserved by f

∴ ~~f is an~~ graphs are isomorphic

39)





Each has 6 vertices ~~edges~~ & 9 edges

Let $f(u_1) = v_1$ $\qquad$ $f(u_4) = v_{\cancel{8}4}$
$f(u_2) = v_2$ $\qquad$ $f(u_5) = v_6$
$f(u_3) = v_3$ $\qquad$ $f(u_6) = v_5$

$$\begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_6 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_6 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

edges are preserved by f

∴ graphs are isomorphic

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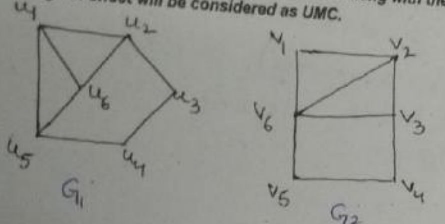
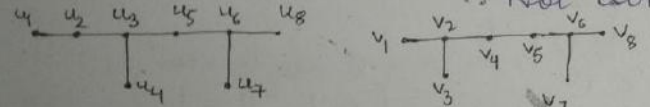
40)



There are 2 vertices u_3 & u_4 with degree 2 in G_1
There are 3 vertices v_1, v_4, v_5 with degree 2 in G_2
∴ Not isomorphic.

41)

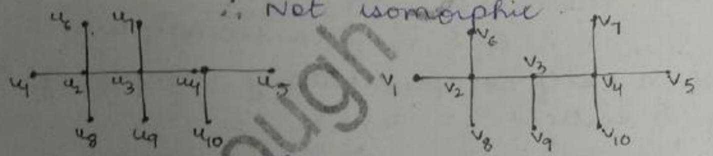


u_2 & u_5 are vertices of degree 2 which are not joined by an edge

v_4 & v_5 are vertices of degree 2 which are adjacent to each other.

∴ Not isomorphic.

42)



u_2 & u_3 are vertices with degree 4 which are adjacent

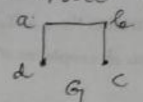
v_2 and v_4 are vertices with degree 4 which are not adjacent.

∴ graphs are not isomorphic.

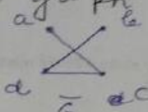
47) Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex
Since isolated vertex has degree 0, it is not adjacent to any vertex ∴ row & column entries are all zeros.

48) Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.
Since isolated vertex is not incident with any ~~vertex~~ edge ∴ all entries are zero.

50) Show that the graph G is self complementary (35)



G



$\overline{G}$

G & $\overline{G}$ have four vertices, 3 edges
There are 2 vertices with degree 1 each &
2 vertices with degree 2 each

Let $f(d) = a$
$f(a) = c$
$f(b) = d$
$f(c) = b$

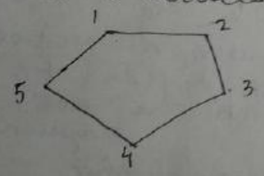
$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad A_{\overline{G}} = \begin{array}{c} \\ c \\ d \\ b \\ a \end{array} \begin{array}{c} \begin{array}{cccc} c & d & b & a \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

G

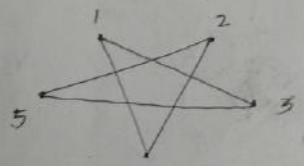
$A_G = A_{\overline{G}}$

$\therefore$ G & $\overline{G}$ are isomorphic.

51) Find a self complementary simple graph with 5 vertices



C_5



$\overline{C_5}$

$f(1) = 1$, $f(2) = 3$, $f(3) = 5$
$f(4) = 2$, $f(5) = 4$

$$A_{C_5} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad A_{\overline{C_5}} = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 3 & 5 & 2 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$A_{C_5} = A_{\overline{C_5}}$

C_5 & $\overline{C_5}$ are isomorphic
Hence C_5 is self complementary

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53) For which integers n is C_n self complementary

If C_n is self complementary

C_n & $\overline{C_n}$ are isomorphic

No of edges should be equal

$$\Rightarrow n = \frac{n(n-1)}{2} - n$$

$$\Rightarrow 1 = \frac{n-1}{2} - 1$$

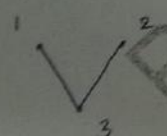
$$\Rightarrow 2 = \frac{n-1}{2} \Rightarrow n = 4+1 = 5$$

Also possibility is C_5

and we have proved C_5 is self complementary.

57) Are the simple graphs with following adjacency matrices isomorphic?

a)
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array} \qquad \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 1 & 0 & 0 \\ c & 1 & 0 & 0 \end{array}$$



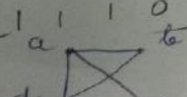
$f(1) = b$, $f(2) = c$, $f(3) = a$.

$$\begin{array}{c|ccc} & b & c & a \\ \hline b & 0 & 0 & 1 \\ c & 0 & 0 & 1 \\ a & 1 & 1 & 0 \end{array}$$

$\therefore$ isomorphic

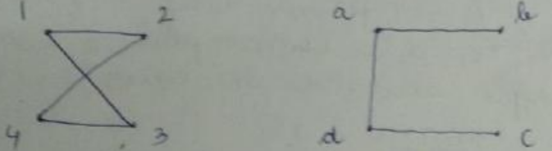
b)
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$

There are 4 edges



There are 5 edges. Not isomorphic

c) $\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$ $\quad$ $\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$



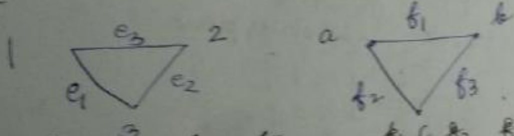
There are 4 edges. There are 3 edges.

Not isomorphic

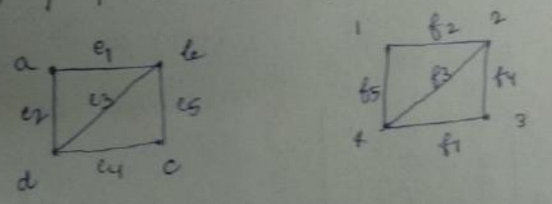
58) Determine whether the graphs without loops with these incidence matrices are isomorphic

a) $\begin{array}{c|c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$, $\begin{array}{c|c} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

They are Isomorphic



b) $\begin{array}{c|c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$, $\begin{array}{c|c} & \begin{array}{ccccc} f_1 & f_2 & f_3 & f_4 & f_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$



They are isomorphic.

Connectivity

__Path__: is a seq of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

Let n be a non negative integer and G an undirected (directed) graph. A __path (walk) of length n__ from u to v in G is a seq of n edges $e_1, \ldots, e_n$ of G s.t e_1 is associated with $\{x_0, x_1\}$ (x_0, x_1), e_2 is associated with $\{x_1, x_2\}$ (x_1, x_2), ... and e_n is associated with $\{x_{n-1}, x_n\}$ (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$.

When graph is simple, path can also be denoted by ~~degree~~ vertex sequence $x_0, x_1, \ldots, x_n$

The path is a __circuit__ (closed walk) (cycle) if it begins and ends at same vertex i.e if $u = v$ and has length greater than zero (walk denoted $x_0, e_1, x_1, e_2, \ldots, e_n, x_n$)

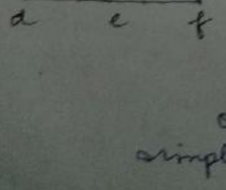
A __path or circuit__ is said to pass through the vertices $x_1, x_2, \ldots, x_{n-1}$ or traverse edges $e_1, e_2, \ldots, e_n$

The path or circuit is __simple__ (trail) if it doesnot contain the same edge more than once.

A __circuit in a graph__ is called __cycle__ in a graph.

* A path of length zero consists of single vertex

__Eg__ what is the length of



~~a d~~ $a, d, c, f, e \rightarrow 4$

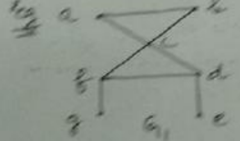
d, e, c, a Not a path

$b, c, f, e, b \rightarrow$ circuit of length 4

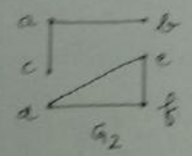
$a, b, e, d, a, b \rightarrow$ path of length 5. It is not simple because it contains the edge $\{a, b\}$ twice.

Def: An <u>undirected graph</u> is called <u>connected</u> if there is a path between every pair of distinct vertices of the graph.

Eg



G_1 is connected since for every pair of distinct vertices there is a path between them

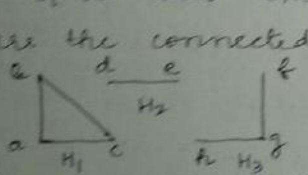


G_2 is not connected since there is no path between c and d.

<u>Theorem</u>: There is a simple path between every pair of distinct vertices of a connected undirected graph

<u>Def</u>: <u>Connected component</u> of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G
A graph G that is not connected has two or more connected components that are disjoint and have G as their union

Eg What are the connected components of graph



Graph is union of three disjoint connected subgraphs H_1, H_2 and H_3. These subgraphs are connected components of H.

* In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these ~~day~~ edges disconnects G.
When a single edge disconnects G, then it is called cut edge
* The removal of a vertex and all edges incident with it produces a subgraph with more connected components than in the original graph. Such vertices are called cut vertices (or articulation pts)

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Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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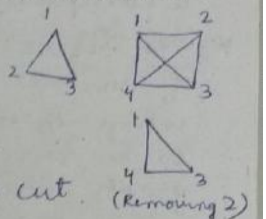
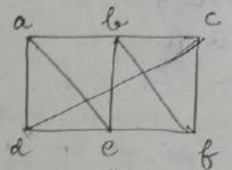
Eg Find cut vertices of $K_n, n \geq 3$.
If we remove any vertex and the incident edges from $K_n, n \geq 3$, we will be left with K_{n-1} which is connected. So $K_n, n \geq 3$ has no cut vertices



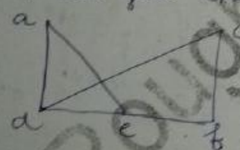
* Connected graphs without cut vertices are called non separable graphs.

* A subset V' of the vertex set V of $G = (V, E)$ is a vertex cut or separating set if $G - V'$ is disconnected

Eg Find a vertex cut in G

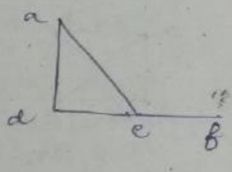


If we remove b & incident edges, we get the subgraph

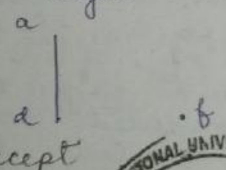


which is connected

Now remove c & incident edges we get a subgraph which is again connected



Now remove e & incident edges we get a subgraph which is disconnected.

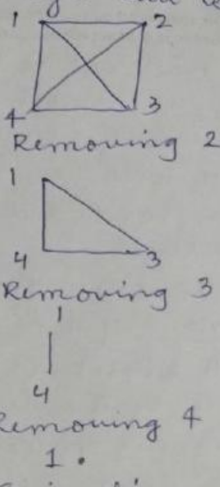


$\therefore$ vertex cut is $\{b, c, e\}$

* Every connected graph, except a complete graph, has a vertex cut.

* Vertex connectivity of a non-complete graph G, denoted by $\kappa(G)$ is the minimum no of vertices in a vertex cut.

* When G is a complete graph, it has no vertex cuts, because removing any subsets of its vertices and all incident edges still leaves a complete graph.



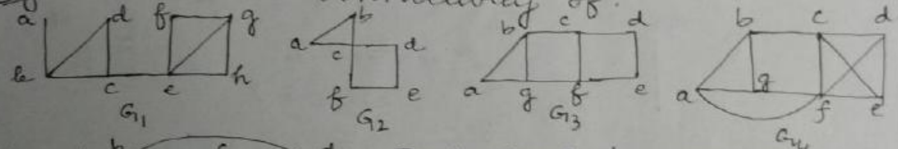
* We define $K(K_n)$ as min no of vertices ~~in a vertex cut when G is complete~~ needed to be removed to produce a graph with a single vertex

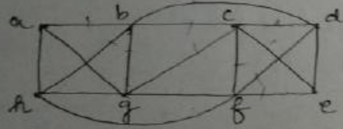
$$K(K_n) = n-1$$

* For every graph G, $K(G)$ is min. no of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex

We have $0 \le K(G) \le n-1$ if G has n vertices, $K(G)=0$ iff G is disconnected or $G = K_1$ and $K(G) = n-1$ iff G is complete.

eg Find vertex connectivity of:



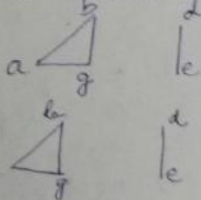


Each graph is connected and has more than one vertex so $K(G_i) > 0 \ \forall i$

G_1 has a cut vertex 'c' so $K(G_1) = 1$

Also G_2 has a cut vertex 'c' so $K(G_2) = 1$

Now in G_3, there are no cut vertices so $K(G_3) > 1$. If we remove c and f and their incident edges then G_3 becomes disconnected so $K(G_3) = 2$. $\{c, f\}$ is vertex cut



In G_4, if we remove c and f then G_4 is disconnected $\{c, f\}$ is vertex cut. $K(G_4) = 2$

In G_5, $\{b, c, f\}$ is vertex cut.

No cut vertices in G_5

$$K(G_5) = 3.$$

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Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

Edge connectivity of a graph G, denoted by $\lambda(G)$, is the minimum no of edges in an edge cut of G.

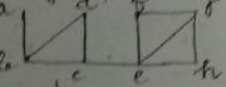
$\lambda(G) = 0$, if G is not connected or if G consists of a single vertex.

If G is a graph with n vertices then $0 \le \lambda(G) \le n-1$

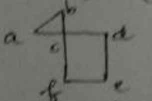
$\lambda(G) = n-1$ where G is a graph with n vertices iff $G = K_n$

so $\lambda(G) \le n-2$ when G is not a complete graph

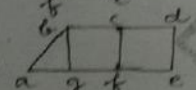
eg Find the edge connectivity of each of the graphs

G_1


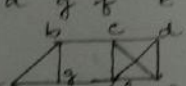
Removal of {c,e} makes graph disconnected so $\lambda(G_1) = 1$

G_2


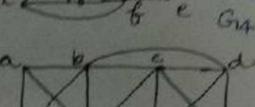
There are no cut edges. Removal of {a, c} & {b, c} leaves graph disconnected so $\lambda(G_2) = 2$

G_3


There are no cut edges. Removal of {a, b} and {a, g} leaves graph disconnected so $\lambda(G_3) = 2$

G_4


Here, no cut edges. Removal of 3 ~~vertices~~ edges {a, b}, {b, g} and {b, c} leaves G_4 disconnected. so $\lambda(G_4) = 3$

G_5


Here, no cut edges. Removal of 3 edges {e, f}, {c, e}, {d, e} leaves graph disconnected so $\lambda(G_5) = 3$

* $\kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v)$

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Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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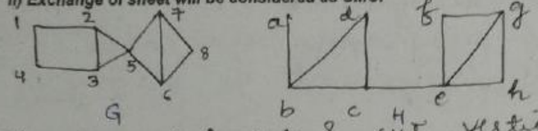
eg



Identify cut sets & cut vertices

$\{(2,5),(3,5)\}$ is a cut set in G

$\{(5,7),(5,6)\}$ is a cut set in G

$\{(1,2),(2,3),(3,5)\}$ is also a cut set in G

If we consider $\{(1,2),(2,3),(3,5),(2,5)\}$, then it is not a cut set because a proper subset of this is a cut set (cocycle)

Cut vertex → 5

Now in H,

cut vertices → b, c, e

Cut edges ~~set~~ → $\{c,e\}$ and $\{a,b\}$

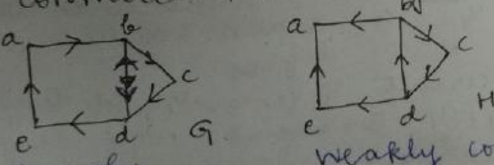
cut sets. $\{\{c,e\}\}$, $\{\{a,b\}\}$, $\{\{e,h\},\{h,g\}\}$, $\{\{f,g\},\{f,e\}\}$, ..

<u>Connectedness in Directed graphs</u>

<u>Def</u> : A directed graph is <u>strongly connected</u> if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

<u>Def</u> : A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph

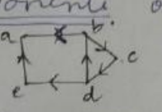
<u>Eg</u> Are the following graphs strongly connected? Are they weakly connected?



strongly connected
Hence, weakly connected.

weakly connected
There is a path from b to a but not from a to b ∴ Not strongly connected

* The subgraphs of directed graph G that strongly connected but not contained in larger strongly connected subgraphs i.e maximal strongly connected subgraphs are called strongly connected components or strong components of G.

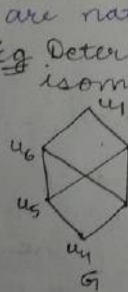
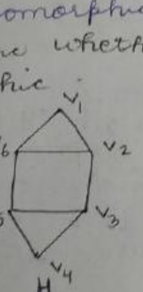
*Eg 

strong components are vertex a; ~~{a}, {~~ vertex e; & graph consisting of vertices b, c, d & edges (b, c), (c, d), (d, b)

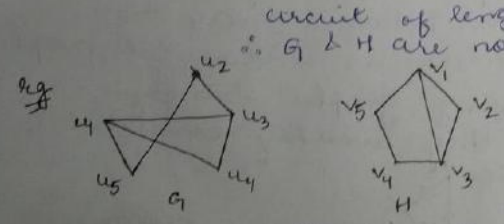
Paths & Isomorphism

The existence of a simple circuit of a particular length k, $k > 2$, $k \in Z$ is a useful invariant that can be used to show that two graphs are not isomorphic.

Eg Determine whether graphs G & H are isomorphic.

They have 6 vertices & 8 edges. Degree of 2 vertices is 2 and degree of 4 vertices is 3.
H has a simple circuit of length 3 (v_1, v_2, v_6, v_1) but G doesnot have a circuit of length 3.
$\therefore$ G & H are not isomorphic.

Eg 

Both have 5 vertices & 6 edges. Both have 3 vertices with degree 2 and 2 vertices with degree 3.

H has a simple circuit v_1, v_2, v_3, v_1 of length 3 & G has u_1, u_3, u_4, u_1 a circuit with length 3
H has simple circuit v_1, v_3, v_4, v_5, v_1 of length 4 and G has u_1, u_3, u_2, u_5, u_1 of length 4.
H has $v_1, v_2, v_3, v_4, v_5, v_1$ of length 5 and G has $u_1, u_5, u_2, u_3, u_4, u_1$ of length 5.
Because all invariants agree, G & H may be

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isomorphic.

Let $f(u_4) = v_2$ $\quad f(u_2) = v_4$
$f(u_1) = v_1$ $\quad f(u_5) = v_5$
$f(u_3) = v_3$

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad A_H = \begin{array}{c} \\ v_1 \\ v_4 \\ v_3 \\ v_2 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_4 & v_3 & v_2 & v_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

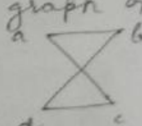
$A_G = A_H$.
$\therefore$ G & H are isomorphic.

Counting paths between vertices

The no of paths between 2 vertices in a graph can be determined using its adjacency matrix.

Theorem:- Let G be a graph with adjacency matrix A w.r.t the ordering $1, 2, \dots, n$ (with directed or undirected edges, with multiple edges and loops allowed). The no. of different paths of length r from i to j where r is a positive integer equals the $(i, j)^{th}$ entry of A^r.

Eg How many paths of length four are there from a to d in simple graph G with respect to ordering a, b, c, d the adjacency matrix is



$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

No. of paths of length 4 ~~are~~ is from a to d is $(1, 4)$th entry of A^4.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 & \boxed{8} \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

there are 8 paths from a to d.

* Suppose that $G = (V, E)$ is a directed graph. A vertex $w \in V$ is reachable from a vertex $v \in V$ if there is a directed path from v to w. The vertices v and w are <u>mutually reachable</u> if there are both a directed path from v to w and a directed path from w to v in G.

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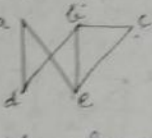
1) Does each of these lists of vertices form a path in the following graph? which paths are simple? which are circuits? what are the lengths of those that are paths?



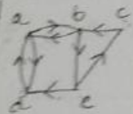
a) a, e, b, c, b — Path, Not simple, Not circuit, length 4

b) a, e, a, d, b, c, a — Not path

c) e, b, a, d, b, e — Not a path

d) c, b, d, a, e, c — Path, simple, circuit, length 5

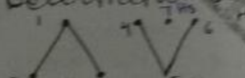
2) Does each of these lists of vertices form a path in the following graph? which paths are simple? which are circuits? what are the lengths of those that are paths?



	Path	Simple	Circuit	length
a) a, b, e, c, b	Yes	Yes	No	4
b) a, d, a, d, a	Yes	NO	Yes	4
c) a, d, b, e, a	NO	-	-	-
d) a, b, e, c, b, d, a	NO	-	-	-

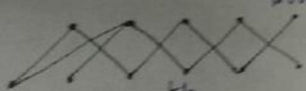
3) Determine whether the given graph is connected. How many connected components are there?

3) & 6)



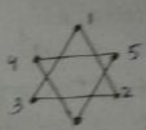
NO, there is no path between 3 and 4
~~Two~~ Three connected components H_1 & H_2 & H_3

4) & 6)



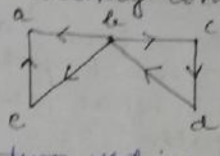
Yes, there is a path ~~through~~ between any two vertices of the graph
One connected component H_3

5) & 6)

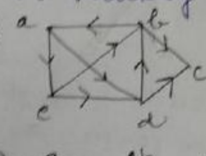


No, there is no path joining 1 to 4.
Two connected components H_4 with vertices 1, 2, 3 & edges {1,2}, {2,3}, {1,3} and H_5 with vertices 4, 5, 6 & edges {4,5}, {5,6}, {6,4}

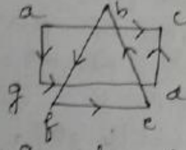
11) Determine whether each of these graphs strongly connected and if not, whether is weakly connected.

a) 

There is a path from b to a but not from a to b
∴ Not strongly connected.
There is a path between every two vertices in underlying undirected graph
∴ Weakly connected.

b) 

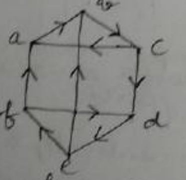
There is a path from b to c but not from c to b. ∴
Not strongly connected
But weakly connected

c) 

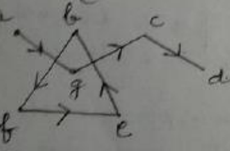
Neither strongly nor weakly connected. No path between a and b in any direction.

12) a) 

There is a path from d to c but not from c to d.
Not strongly connected.
But it is weakly connected.

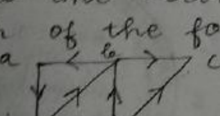
b) 

There is a path from one vertex to the other vertex of graph in both direction
∴ Strongly connected. Hence, weakly connected.

c) 

There is no path from a to b or b to a so it is neither strongly nor weakly connected.

14) Find the strongly connected components of each of the following :-

a) 

Strong component is the graph with vertices a, b and e & edges (a,e), (e,b), (b,a) ; vertex c; vertex d

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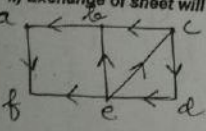
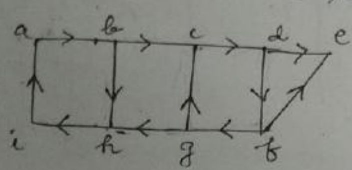
b)

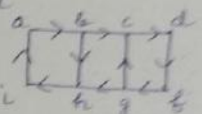


Strong component is graph with vertices c, d and e and edges (c, d), (d, e), (e, c) ; vertex f ; vertex a ; vertex b.

c)

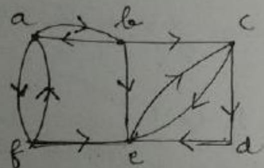


graph with a, b, c, d, f, g, h, i as vertices and edges represented in graph (1).

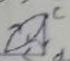


is strong component
Vertex e.

15) a)

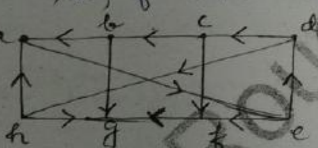


strong component graph with c, e, d vertices & edges (c, d), (d, e), (e, c) (c, e)

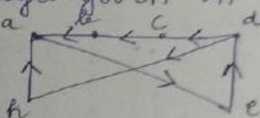


strong component graph with vertices a, b, f and edges (a, f), (f, a), (a, b), (b, a)

b)

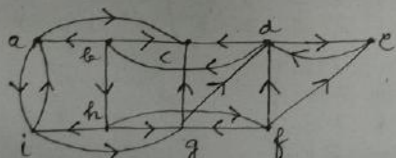


graph with a, b, c, d, e and h as vertices and edges given in graph



& {f} &
{g}

c)



{a, b, d, e, f, g, h, i}
and {c}

16) Show that if $G = (V, E)$ is a directed graph and u, v and w are vertices in V for which u and v are mutually reachable and v and w are mutually reachable then u and w are mutually reachable.

Sol There is a path from u to v and v to w so there is a path from u to w and then
Now there is a path from w to v and then

from v to u so there is a path from v to u so u and v are mutually reachable

19) Find the no of paths of length n between two different vertices in K_4 if n is

a) 2

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

There are 3 paths between any two different vertices of length 2

b) 3

$$A^3 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$$

There are 7 paths b/w any 2 different vertices of length 3

c) 4

$$A^4 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix}$$

There are 20 paths b/w any 2 different vertices of length 4

d) 5

$$A^5 = \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 60 & 61 & 61 & 61 \\ 61 & 60 & 61 & 61 \\ 61 & 61 & 60 & 61 \\ 61 & 61 & 61 & 60 \end{bmatrix}$$

There are 61 paths b/w any 2 different vertices of length 5

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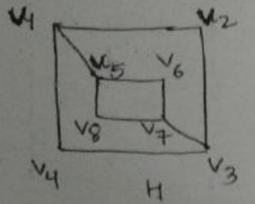
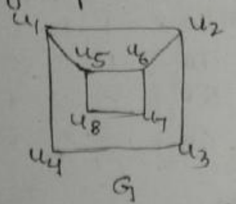
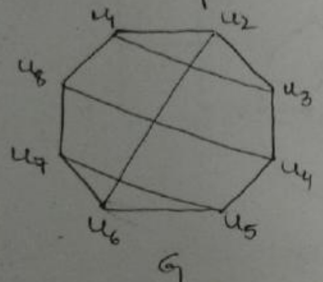
20) Are the graphs isomorphic or not?





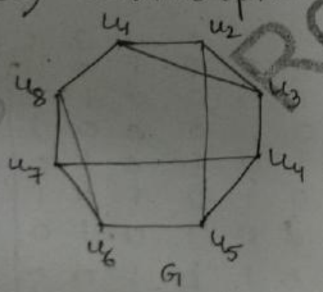
In G, there are ~~two~~ three 4-cycles but in H, there ~~is only~~ are two ~~one~~ 4-cycles ∴ Not isomorphic

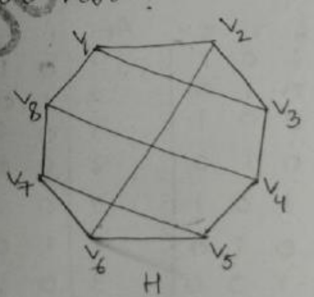
21) Isomorphic or not?



In G, there are 2 three cycles but in H, there is no 3-cycle ~~so~~ G $\ncong$ H.

22) Isomorphic or not?





8 vertices
12 edges
Degree all 3.
There are ~~two~~ ~~three~~ 3 cycles.
two 4-cycle.
~~two~~ four 5-cycles.
one 6-cycle
two 7-cycles
one 8-cycle

$f(u_1) = v_2$
$f(u_2) = v_1$
$f(u_3) = v_3$
$f(u_4) = v_4$
$f(u_5) = v_8$
$f(u_6) = \cancel{v_6}\ v_7\ v_5/v_7$
$f(u_7) = \cancel{v_4}\ v_5$
$f(u_8) = v_6$

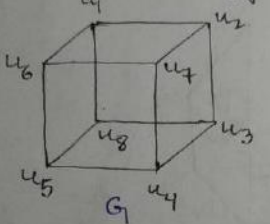
$$A_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_H = \begin{array}{c} \\ 2 \\ 1 \\ 3 \\ 4 \\ 8 \\ 7 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccccc} 2 & 1 & 3 & 4 & 8 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

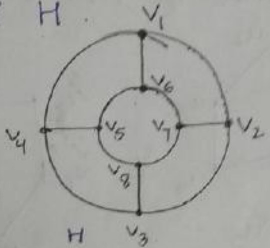
$A_G = A_H$

$\therefore G \cong H$.

23)



G



H

$f(u_1) - v_1$
$f(u_2) - v_2$
$f(u_3) = v_7$
$f(u_4) = v_8$

$f(u_5) = v_5$
$f(u_6) = v_4$
$f(u_7) = v_3$
$f(u_8) = v_6$

$$A_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A_H = \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \\ 5 \\ 4 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 7 & 8 & 5 & 4 & 3 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$A_G = A_H$

$\therefore$ G & H are isomorphic.

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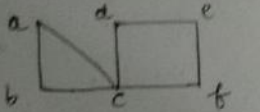
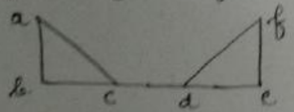
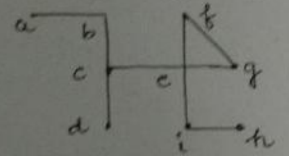
Find all the cut vertices and cut edges

		Cut vertices	Cut edges
31) 2,3,4)		c.	No
32) 2,3,4)		c, d	{c, d}
33) 2,3,4)		b, c, e, i	{a, b}, {b, c}, {c, d}, {c, e}, {e, i}, {h, i}

OMR Sheet No.__________ Registration No. 54

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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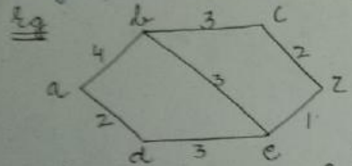
Shortest Path Problems

Graphs that have a number assigned to each edge are called weighted graphs.

Eg



length of path in weighted graph is the sum of the weight of the edges of this path.

What is the length of a shortest path between a and z in the weighted graph? (Dijkstra's Algorithm)

a — b = 4 ✓
a — d = 2 ✓

d — e = 5 ✓

b — c = 7 ✗
b — e = 7 ✗

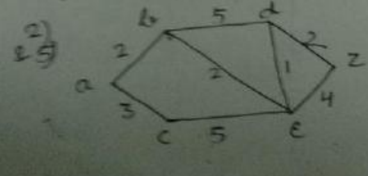
e — z = 6

If we go from c to z then length is 9
So shortest length is 6 along the path "

Shortest Path a — d — e — z = 6

Ques Find the shortest distance between ~~states~~ a and z

2)
25



a — b = 2 ✓
a — c = 3 ✓

b — d = 7 ×
b — e = 4 ✓

c — e = 8 ×

e — d = 5 ✓
e — z = 8 ×

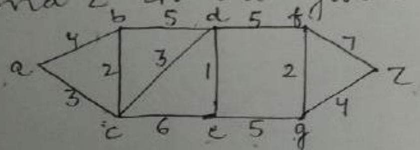
d — z = 7 ✓

So, shortest distance from a to z is 7.

Shortest path is a → b → e → d → z

Find the length of a shortest path between and z in the given weighted graph

3) 4 5)



a — b = 4 ✓
a — c = 3 ✓

c — b = 5 ×
c — d = 6 ✓
c — e = 9 ×

b — d = 9 ×

d — e = 7 ✓
d — f = 11 ✓

e — g = 12 ✓

f — g = 13 ×
f — z = 18 ×

g — z = 16 ✓

So shortest distance is 16.

Shortest path is

a
|
c
|
d
|
e
|
g
|
z

6) Find length of shortest path b/w
a) a and d - 6
b) a and f - 11

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Registration No. 56 26

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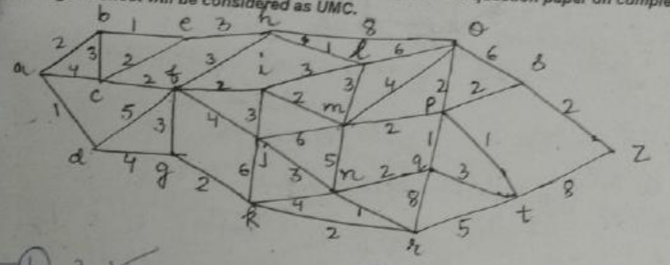


a — b = 2 ✓
a — c = 4 ✓
a — d = 1 ✓

d — f = 6 ✓
d — g = 5 ✓

b — c = 5 ✗
b — e = 3 ✓

e — c = 5 ✗
e — h = 6 ✓

c — f = 6 ✗

g — f = 8 ✗
g — k = 7 ✓

f — h = 9 ✗
f — i = 8 ✓
f — j = 10 ✓

h — l = 9 7 ✓
h — o = 14 ✗

k — j = 13 ✗
k — n = 11 ✗
k — r = 9 ✓

i — j = 11 ✗
i — l = 11 ✗
i — m = 10 ✓

l — m = 10 ✗
l — o = 13 ✓

r — n = 10 ✓
r — q = 17 ✗
r — t = 14 ✗

j — m = 16 ✗
j — n = 13 ✗

m — n = 15 ✗
m — p = 12 ✓
m — o = 14 ✗

n — ~~r = 11 ✓~~
n — q = 12 ✓

p — o - 14 ✗
p — q = 13 ✗
p — t = 13 ✓
p — s = 14 ✓

q — t = 15 ✗

o — s = 19 ✗

t — z = 21 ✗

s — z = 16 ✓

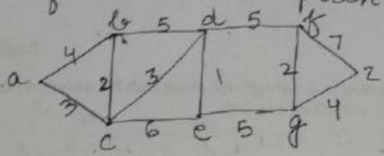
Shortest distance = 16

Shortest path

a – d – f – i – m – p – s – z

2&4) 6) Find the length of a shortest path b/w

c) c and f



57

c — a = 3 ✓
b = 2 ✓
d = 3 ✓
e = 6 ×

b — a = 6 ×
d = 7 ×

a — ×

d — e = 4 ✓
f = 8 ✓

e — g = 9

f — g = 10
z = 15

So length of shortest path b/w c & f is 8

Path is c - d - f

d) b and z

b — a = 4 ✓
c = 2 ✓
d = 5 ✓

c — a = 5 ×
d = 5 ×
e = 8 ×

a — ×

d — e = 6 ✓
f = 10 ✓

e — g = 11 ✓

f — g = 12 ×
z = 17 ×

g — z = 15 ✓

so shortest path is
b - d - e - g - z
length is 15

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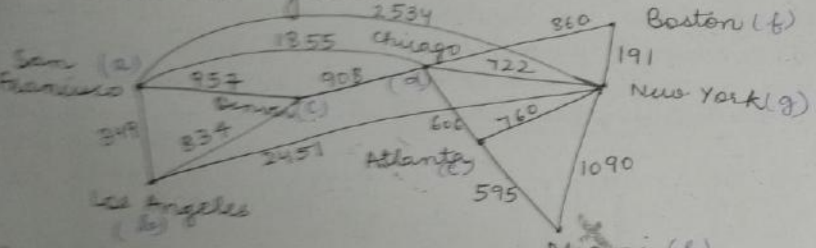
Registration No. 58

This sheet must be submitted to the invigilator along with the question paper on completion of examination. Exchange of sheet will be considered as UMC.

Find a shortest path (in mileage) between each of the following pairs of cities in airline system



a) New York and Los Angeles
g and b

b — a = 349 ✓
c = 834 ✓
g = 2451 ✓

a — g = 2883 ×
d = 2204 ×
c = 1306

c — d = 1742 ✓

d — f = 2602 ✓
g = 2464 ×
e = 2348 ✓

e — g = 3108 ×
h = 2943 ✓

g — h = 3541 ×

~~f — g = 2793~~ ✓

2348 + 760 = 3108

834 + 908 = 1742

2534 + 349 = 2883

2348 + 595 = 2943

1742 + 860 = 2602

1855 + 349 = 2204

2451 + 1090 = 3541

1742 + 722 = 2464

957 + 349 = 1306

2602 + 191 = 2793

1742 + 606 = 2348

length of shortest ~~distance~~ path is
2451
$b \to g$
Los Angeles → New York

d) Miami and Los Angeles
h and b
length of shortest path is 2943

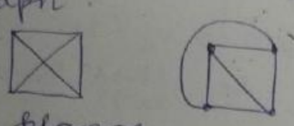
b — c — d — e → h
Los Angeles — Denver — Chicago — Atlanta — Miami

Rough Sheet

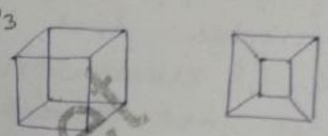
Planar graphs

A graph is called planar if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a pt other than their common endpoint). Such a drawing is called a planar representation of the graph.

K_4



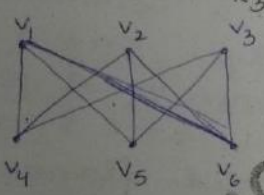
Q_3



K_4 is planar

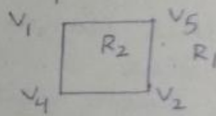
Q_3 is planar.

eg Show that $K_{3,3}$ is non planar.

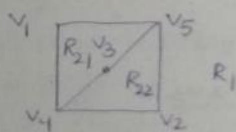


Here, we always have v_1 and v_2 connected with both v_5 and v_4. These 4 edges form a closed curve that splits the plane into two regions R_1 and R_2.

Now vertex v_3 is either in R_1 or R_2



When v_3 is in R_2, the edges b/w v_3 & v_4 and v_3 and v_5 separate R_2 into subregions R_{21} and R_{22}



Now we have to place V_6

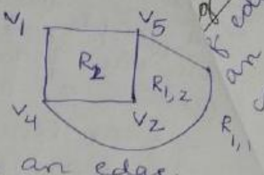
If we place V_6 in R_1 then the edge joining v_3 & v_6 crosses an edge

If we place V_6 in R_{21} then edge b/w v_2 & v_6 crosses an edge

If we place v_6 in R_{22}, then edge b/w crosses an edge

So there is no way to place the vertex without forcing a crossing.

Similarly if v_3 is in R_1 then edges $\{v_5, v_3\}$ & $\{v_4, v_3\}$ divides R_1 into R_{11} & R_{12}



Now if we try to place v_6 in R_2, then $\{v_6, v_3\}$ crosses an edge
If we place v_6 in $R_{1,1}$ then $\{v_2, v_6\}$ crosses an edge. And if we place v_6 in $R_{1,2}$ then $\{v_1, v_6\}$ crosses an edge. So this case is also not possible.

Hence $K_{3,3}$ is not planar.

Euler's Formula: Let G be a connected planar simple graph with e edges and v vertices. Let r be the no. of regions in a planar representation of G. Then $r = e - v + 2$.

Eg Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Sol Here $v = 20$

Each vertex has degree 3.

$$e = \frac{20 \times 3}{2} = 30.$$

Now $r = e - v + 2 = 30 - 20 + 2 = 12$

Corollary 1: If G is a connected planar simple graph with e edges and v vertices, $v \geq 3$ then $e \leq 3v - 6$

Corollary 2: If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Corollary 3: If a connected planar simple graph has e edges and v vertices, $v \geq 3$ and no circuits of length three then $e \leq 2v - 4$.

et No.________ Registration No. 61 27

i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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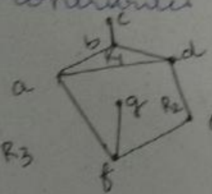
Degree of a region is defined to be the no. of edges on the boundary of the region when an edge occurs twice on the boundary, it contributes two to the degree



$deg(R_1) = 3$
$deg(R_2) = 6$
$deg(R_3) = 7$

eg Show that K_5 is non planar.
K_5 is connected simple graph.
It has 5 vertices and $\frac{5\times4}{2} = 10$ edges

Now if K_5 is planar
then by corollary 1, $e \le 3v - 6$
$\Rightarrow 10 \le 3\times5 - 6$
$10 \le 15 - 6 = 9$
which is not true

Hence K_5 is not planar.

eg Show that $K_{3,3}$ is non planar
$K_{3,3}$ is connected simple graph.

It has 9 edges & 6 vertices.
Moreover, it has no circuit of length three.
If it is planar then by corollary 3,
$e \le 2v - 4$
$\Rightarrow 9 \le 2\times6 - 4$
$= 12 - 4 = 8$ which is not true

Hence $K_{3,3}$ is non planar.

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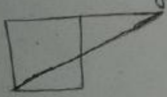
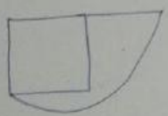
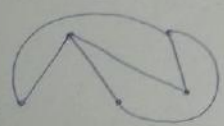
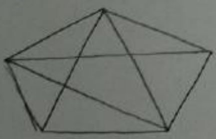
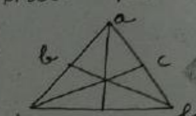
In Exercise 2-4 draw the given planar graph without any crossings.

2)  

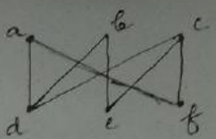
3)  

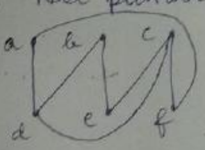
4)  

In Exercises 5-9 determine whether the given graph is planar. If so, draw it so that no edges cross.

5) 

$e = 9$
$v = 6$
$9 \nleq 2v - 4 = 2\times 12 - 4 = 8$
$e \nleq 2v - 4$
$\therefore$ Not planar

6) 



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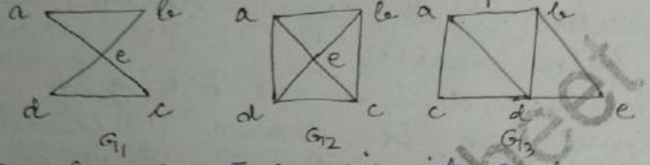
Euler and Hamilton Paths

Def : An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Eg Which of the undirected graphs have an Euler circuit? Of those that donot, which have an Euler path?

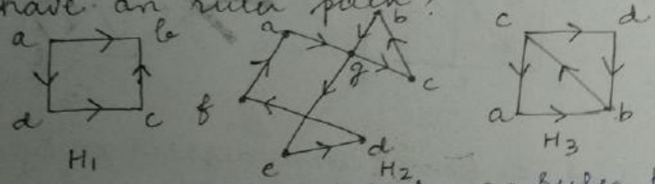


G_1 has an Euler circuit a, b, e, d, c, e, a

G_2 doesnot have an Euler circuit. It doesnot have an Euler path.

G_3 doesnot have an Euler circuit but it has an Euler path a, b, d, a, c, d, e, b

Eg which of the directed graphs have an Euler circuit? Of those that donot, which have an Euler path?



In H_1, no Euler circuit, no Euler path.

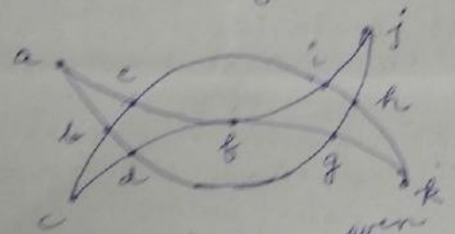
In H_2, there is an Euler circuit a, g, c, b, g, e, d, f, a.

In H_3, no Euler circuit but there is an Euler path c, a, b, c, d, b

Necessary and sufficient conditions (58) for Euler circuits and paths

Theorem: A connected multigraph with at least two vertices has an Euler circuit iff each of its vertices has even degree

Eg



Does it have an Euler circuit?

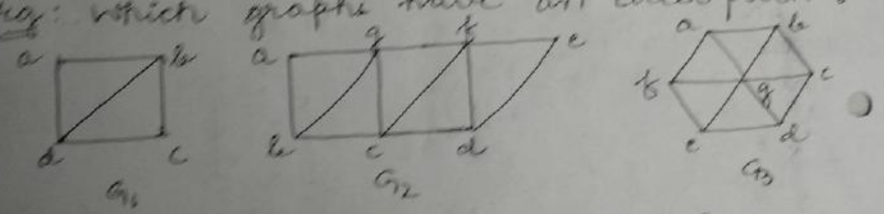
Every vertex is of even degree. Hence, it has an Euler circuit.

a, e, f, i, j, h, g, d, c, b, e, i, h, k, g, f, d, b, a

This figure can be drawn without lifting the pen or retracing part of figure

Theorem: A connected multigraph has an Euler path but not an Euler circuit iff it has exactly two vertices of odd degree.

Eg: Which graphs have an Euler path?



In G_1, a and c have even degree (2)
b and d have odd degree (3)
so there is an Euler path d, a, b, c, d, b

In G_2, a (deg 2), g (deg 4), f (deg 4), e (deg 2), c (deg 4) have even degrees and d (deg 3) and b (deg 3) have odd degree. So, there is an Euler path b, a, g, b, c, g, f, c, d, f, e, d

In G_3, a, b, c, d, e, f all have odd deg 3 and g has even degree 6 so, there is no Euler path.

R Sheet No.__________ Registration No. 65

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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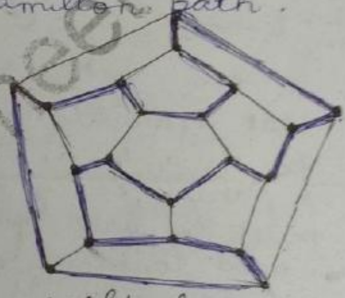
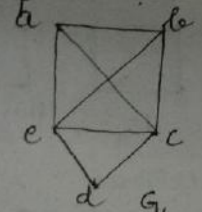
Def A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit. That is the simple path $x_0, x_1, \dots, x_{n-1}, x_n$ in the graph $G=(V,E)$ is a Hamilton path if $V=\{x_0, x_1, \dots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \le i < j \le n$ and the simple circuit $x_0, x_1, \dots, x_{n-1}, x_n, x_0$ (with $n>0$) is a Hamilton circuit if $x_0, x_1, \dots, x_{n-1}, x_n$ is a Hamilton path.

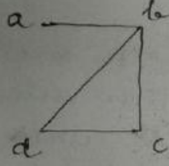
Eg Is there a Hamilton circuit in the graph? Yes, it is shown in the graph.



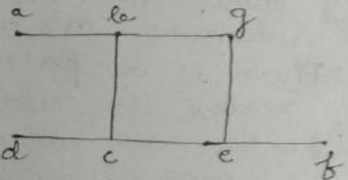
Eg Which of the simple graphs have a Hamilton circuit, or if not, a Hamilton path?



G_1



G_2



G_3

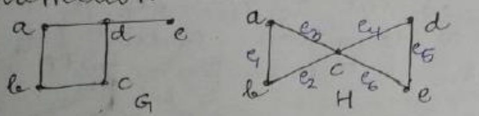
G_1 has hamilton circuit a, b, c, d, e, a

There is no hamilton circle in G_2 but there is a hamilton path a, b, c, d

G_3 has neither a Hamilton path nor a Hamilton circuit.

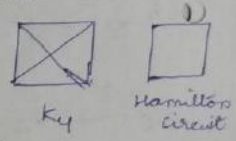
* A graph with vertex of degree one cannot have Hamilton ~~circle~~ circuit
* If a vertex in the graph has degree two then both edges that are incident with this vertex must be part of any Hamilton circuit. When a Hamilton circuit is constructed and this circuit has passed through a vertex then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration.

Eg Show that neither of these graphs has a Hamilton circuit.



G has vertex e of degree 1, so it has not Hamilton circuit.
In H, a, b, d, e are vertices of degree 2, then every edge incident with them must be part of Hamilton circuit i.e circuit would contain $e_1, e_2, e_3, e_4, e_5, e_6$ which is not possible as the circuit also would contain four edges incident with c.

Eg Show that K_n has a Hamilton circuit whenever $n \geq 3$



There is a path from one vertex to every other vertex in K_n. So, we start with any vertex, we can visit the vertices through edges in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. So K_n has a Hamilton circuit.

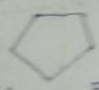
MR Sheet No.__________ Registration No. 67

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

Dirac's theorem:- If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is atleast $n/2$ then G has a Hamilton circuit.

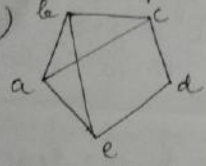
Ore's theorem:- If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

* These two theorems give sufficient conditions to have Hamilton circuit. They donot give any necessary condition. For eg: C_5 has a Hamilton circuit. It doesnot satisfy Dirac's theorem as degree of each vertex is 2 which is less than $5/2$

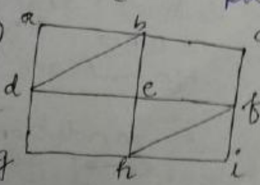


It doesnot satisfy Ore's theorem as $deg(u) + deg(v) = 4$ for every pair of non adjacent vertices u & v in C_5

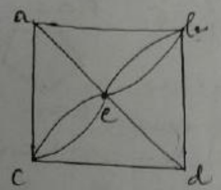
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1) 

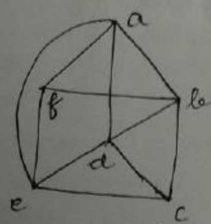
Each vertex doesnot have an even degree so NO Euler circuit. There are four vertices with odd degree so there doesnot exist an Euler path

2) 

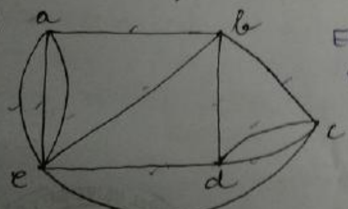
Each vertex has even degree.
a, b, c, f, i, h, g, d, e, h, f, e, b, d, a

3) 

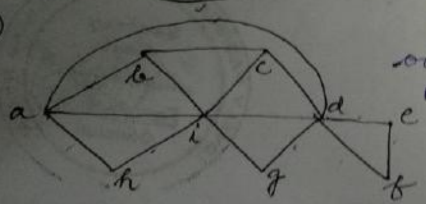
Each vertex doesnot have even degree. No Euler circuit.
There are two vertices of odd degree So there is an Euler path.
a, b, d, e, b, e, c, e, a, c, d

4) 

Each vertex doesnot have even degree. No Euler circuit.
There are two vertices of odd degree. So there is an Euler path
c, e, a, b, c, d, b, f, a, d, e, f

5) 

Every vertex has even degree So there is an Euler circuit
a, b, c, e, a, e, d, c, d, b, e, a

8) 

b and c are vertices of odd degree. No Euler circuit But there is an Euler path
b, a, d, e, f, d, g, i, h, a, i, d, c, i, b, c

MR Sheet No.__________

Registration No. 69

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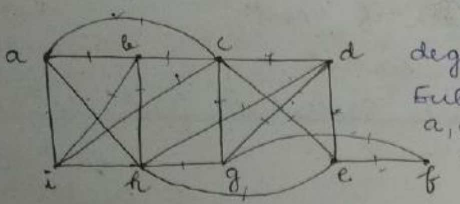
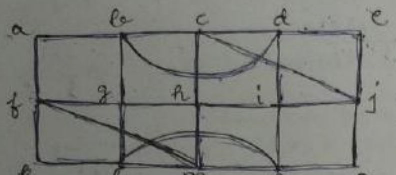
ii) Exchange of sheet will be considered as UMC.

7)



Every vertex has even degree. so there is an Euler circuit.
a, c, d, e, f, g, d, h, g, c, b, h, e, c, i, b, a, h, i, a

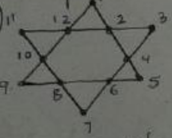
8)



Every vertex has even degree so there is an Euler circuit.
a, b, c, d, b, g, h, c, j, e, d, i, j, o, n, l, h, m, n, e, m, f, g, e, k, f, a.

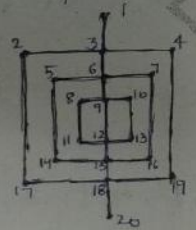
Determine whether the picture can be drawn with a pencil in a cts motion without lifting the pencil or retracing the part of the picture.

13)



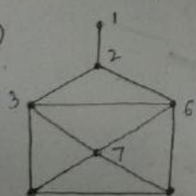
Every vertex has even degree. So there is an Euler circuit Hence, the picture can be drawn in cts motion 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 2, 4, 6, 8, 10, 12, 1.

9)



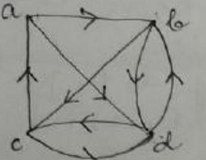
There are two vertices with odd degree there is an Euler path. Hence, the picture can be drawn
1, 3, 2, 17, 18, 19, 4, 3, 6, 5, 14, 15, 16, 7, 6, 9, 8, 11, 12, 13, 10, 9, 12, 15, 18, 20

15)



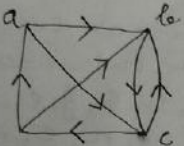
There are 4 vertices of odd deg. There is no Euler path Hence we can't drawn the picture in a cts motion.

Determine whether the directed graph (70) has a Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if exist

18)


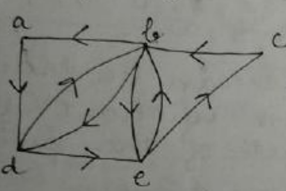
No Euler circuit

Euler path a, b, d, b, c, d, c, a, d

19)


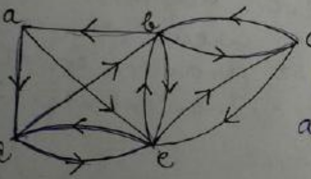
No Euler circuit

No Euler path

20)


Euler circuit

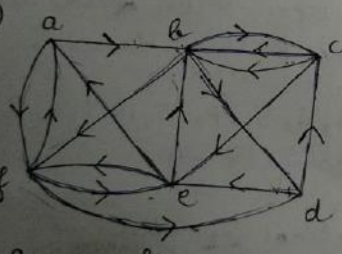
b, a, d, b, d, e, b, e, c, b

21)


No Euler circuit

Euler path -

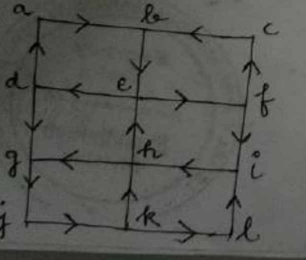
a, d, e, d, b, c, b, e, b, a, e, c, e.

22)


No Euler circuit

Euler path

b, c, b, d, c, e, f, e, b, f, d, e, a, f, a, b, c.

23)


No Euler circuit

No Euler path

MR Sheet No.__________

Registration No. 71

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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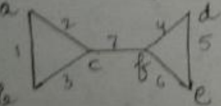
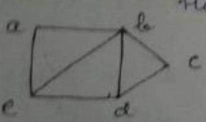
Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit If it doesnot, give an argument to show why no such circuit exists.

30) & 37)



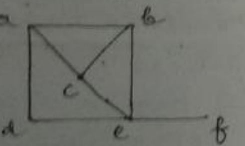
If Hamilton circuit exists, edge 1, 2, 3, 4, 5, 6 has to be there since a, b, d, e are vertices of degree 2 so incident edges will be there in Hamilton circuit
Such a hamilton circuit is not possible.
Hamilton path – a, b, c, f, d, e

31) & 38)



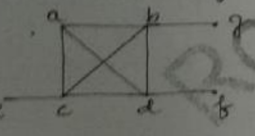
There exists a Hamilton circuit
a, b, c, d, e, a.
Hamilton path a, b, c, d, e

32) & 39)



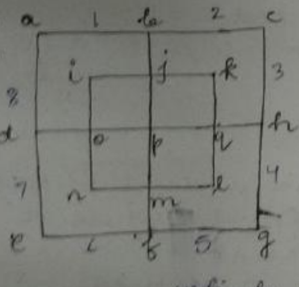
There is a vertex f with degree 1 so there is no Hamilton circuit.
Hamilton path – d, a, c, b, e, f

33) & 40)



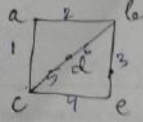
There is a vertex of degree 1 (e and g) so there is no Hamilton circuit.
No Hamilton path.

34) & 41)



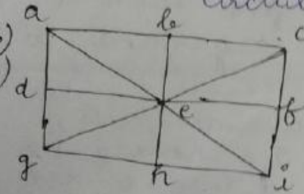
If Hamilton circuit exists, then it will consist of edges 1, 2, 3, 4, 5, 6, 7, 8 ∵ a, c, g, e are vertices of deg 2 so Hamilton circuit would consist of incident ~~vertices~~ edges to these vertices
which can be possible as repetition of vertices is not possible.
So No Hamilton circuit.
No Hamilton path

35) & 42)



Since d, a, e are of (T) deg 2 so if Hamilton circuit exists it would consist of every edge graph which is not possible for Hamilton circuit so No Hamilton circuit.

36) & 43)



Hamilton path – a, b, e, c, d

~~b, d, f, h are verte~~

Hamilton circuit is

a, b, c, f, i, h, g, d, e, a

Hamilton path

a, b, c, f, i, h, g, d, e

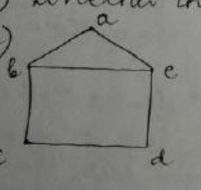
47) For each of these graphs, determine

i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit

ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit.

iii) whether the graph has a Hamilton circuit.

a)



i) Here, we have 5 vertices

If every vertex has degree atleast $5/2$ then we can apply Dirac's which is not so as $deg(a) = 2$.

So Dirac's thm can't be used.

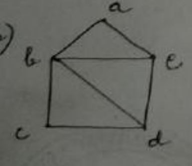
ii) Here a and c are non-adjacent

$deg(a) + deg(c) = 2 + 2 = 4 < 5$

So Ore's thm can't be used.

iii) The graph has a Hamilton circuit a, e, d, c, b, a

b)



i) 5 vertices

$deg(a) = 2 < 5/2$

So Dirac's theorem can't be used

ii) $deg(a) + deg(c) = 4 < 5$ where a and c are non adjacent

So Ore's thm can't be used.

iii) The graph a Hamilton circuit a, b, c, d, e, a

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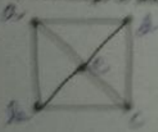
c)


i) Here, we have 5 vertices
degree of each vertex is either 3 or 4 which is greater than $5/2$.
so Dirac's thm can be applied
Hence, it has Hamilton circuit.

ii) For any two non adjacent vertices say a & c
$deg(a) + deg(c) = 6 > 5$
so Ore's theorem can be applied
$\therefore$ it has Hamilton circuit.

iii) Hamilton circuit is a, d, ~~c~~ b, c, a

d)


i) There are 6 vertices & every vertex has degree $3 = 6/2$ so there is a Hamilton circuit by Dirac's thm

ii) For any non adjacent vertices a & b $deg(a) + deg(b) = 6 = n$ so there is a Hamilton circuit by Ore's thm.

iii) Hamilton circuit is 1, 2, 3, 4, 5, 6, 1

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Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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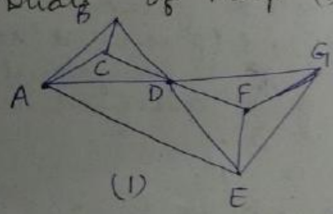
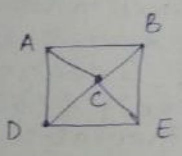
Graph Coloring

Consider the problem of determing the least no. of colours that can be used to color a map so that adjacent regions never have the same color.
Each map in the plane can be represented by graph. Each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one pt are not considered adjacent. The resulting graph is called the dual graph of the map
Any map in the plane has a planar dual graph
Duals of Map (1) & (2)



(1)



The problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color

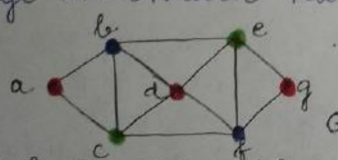
Def: A coloring of a simple graph is the assignment of a color to each vertex of graph so that no two adjacent vertices are assigned the same color.

Def The chromatic number of a graph is the least no of colors needed for a coloring of this graph. The chromatic no. of a graph G is denoted by $\chi(G)$

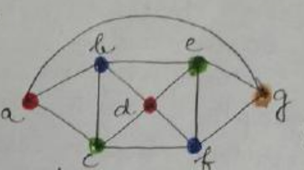
Thm: The Four Color Theorem:- The chromatic number of a planar graph is no greater than four.

* Four colour theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers

eg



G



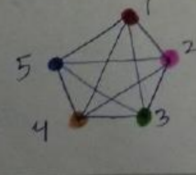
H

What are the chromatic numbers of the graphs G and H ?

The chromatic no of G is atleast 3, because the vertices a, b and c must be assigned different colors. Assign red to a, blue to b, green to c. Now d must be colored red because it is adjacent to b and c. e is adjacent to b and d so it can't be assigned red, blue and e is assigned green.
f is adjacent to c(green), e(green), d(red), so f is assigned blue. g is adjacent to e(green), f(blue), so g is assigned red.
So chromatic number is 3.

The chromatic no of H is 4.

eg What is the chromatic no of K_n ?
Here, every two vertices of this graph are adjacent. Hence, chromatic no is n. $\chi(K_n)=n$



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i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
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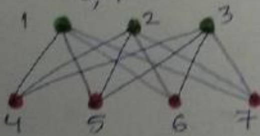
Q3 What is the chromatic no. of the complete bipartite graph $K_{m,n}$ where m and n are positive integers?

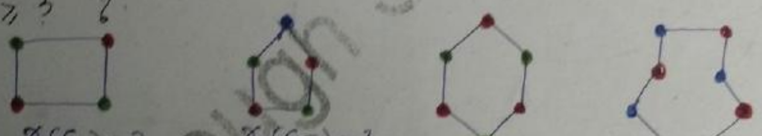
We can color the set of m vertices with one color and set of n vertices with a second color. Because edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color.

$K_{3,4}$



$\chi(K_{3,4}) = 2$

Q4 What is the chromatic number of C_n, $n \geq 3$?

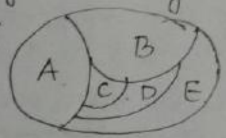


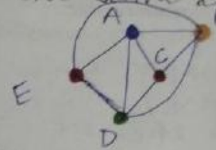
$\chi(C_4) = 2$ $\chi(C_5) = 3$. $\chi(C_6) = 2$ $\chi(C_7) = 3$

$\chi(C_n) = 2$ if n is even positive integer.
$\chi(C_n) = 3$ if n is odd positive integer.

In Exercises 1-4 construct the dual graph for the map shown. Then find the no. of colours needed to color the map so that no two adjacent regions have the same color.

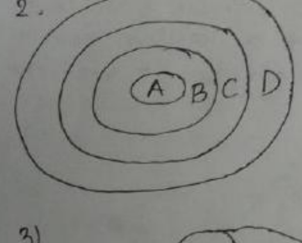
1.

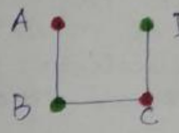




chromatic no - 4

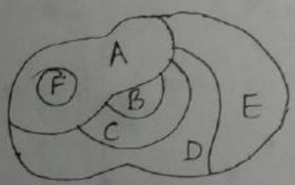
2.

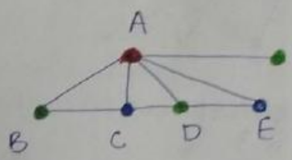




chromatic no - 2

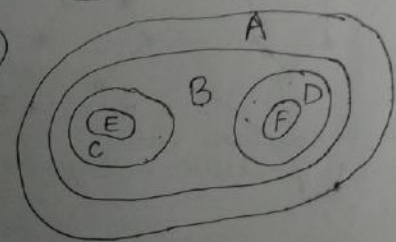
3)

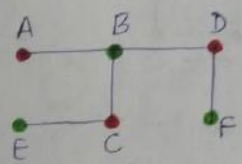




chromatic no - 3

4)



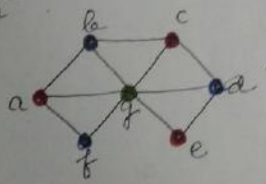


In Exercises 5 - 11 find the chromatic no of the given graph

5)

$\chi(G) = 3$

6)



$\chi(G) = 3$

7)

$\chi(G) = 3$

8)

$\chi(G) = 3$

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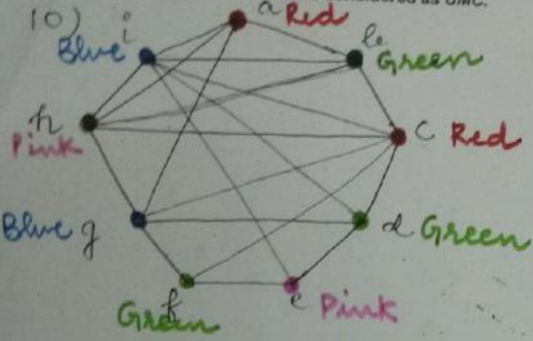
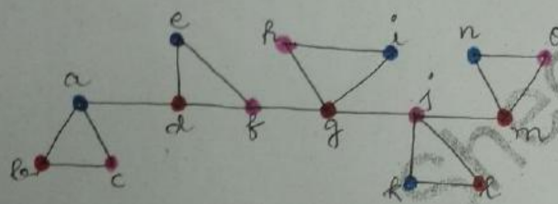
10)



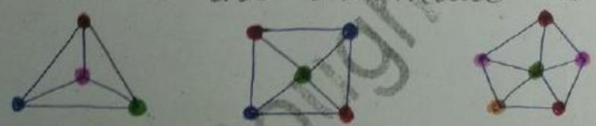
$\chi(G) = 4$.

11)



$\chi(G) = 3$.

15) What is the chromatic no. of W_n?



$\chi(W_3) = 4$ $\chi(W_4) = 3$ $\chi(W_5) = 4$

$$\chi(W_n) = \begin{cases} 3, & n \text{ is even} \\ 4, & n \text{ is odd.} \end{cases}$$